ภาคผนวก

ภาคผนวกที่ 1

ค่าคงที่ทางกายภาพ

Constant and symbol[1]		SI value	Gaussian value
Speed of light in vacuum	c	2.9979246×10^{8} m/s	2.9979246×10^{10} cm/s
Proton charge	e	1.60219×10^{-19} C	
	e'		4.80324×10^{-10} statC
Permittivity of vacuum	ε_0	8.85419×10^{-12} $C^2/N\text{-}m^2$	
Avogadro constant	N_0	6.0220×10^{23} $mole^{-1}$	6.0220×10^{23} $mole^{-1}$
Electron rest mass	m	9.1095×10^{-31} kg	9.1095×10^{-28} g
Proton rest mass	m_p	1.67265×10^{-27} kg	1.67265×10^{-24} g
Neutron rest mass	m_n	1.67495×10^{-27} kg	1.67495×10^{-24} g
Planck constant	h	6.6262×10^{-34} J s	6.6262×10^{-27} erg s
	$\hbar$	1.05459×10^{-34} J s	1.05459×10^{-27} erg s
Faraday constant	F	96485 C/mole	
Permeability of vacuum	μ_0	$4\pi \times 10^{-7}$ N C^{-2} s^2	
Bohr radius	a_0	5.29177×10^{-11} m	0.529177×10^{-8} cm
Bohr magneton	β_e	9.2741×10^{-24} J/T	
Nuclear magneton	β_N	5.05082×10^{-27} J/T	
Electron g value	g_e	2.00231931	2.00231931
Proton g value	g_p	5.58569	5.58569
Gas constant	R	8.314_4 J/mole-K	$8.314_4 \times 10^{7}$ erg/mole-K
Boltzmann constant	k	1.3807×10^{-23} J/K	1.3807×10^{-16} erg/K
Gravitational constant	G	6.67×10^{-11} $m^3/kg\text{-}s^2$	6.67×10^{-8} $cm^3/g\text{-}s^2$

Adapted from E. R. Cohen and B. N. Taylor, *J. Phys. Chem. Ref. Data*, **2**, 663 (1973).

[1] $\hbar = h/2\pi$, $F = N_0 e$, $e' = e/(4\pi\varepsilon_0)^{1/2}$, $a_0 = \hbar^2/me'^2 = 4\pi\varepsilon_0\hbar^2/me^2$, $\beta_e = e\hbar/2m$, $\beta_N = e\hbar/2m_p$, $k = R/N_0$.

ภาคผนวกที่ 2

การเปลี่ยนหน่วยของพลังงานและการเรียกชื่อสัญลักษณ์ที่เป็นอักษรกรีก

การเปลี่ยนหน่วยของพลังงาน

		J	erg	eV	cm^{-1}	hartree	kcal/mole
1 J	=	1	10^{7}	6.2415×10^{18}	5.0340×10^{22}	2.2937×10^{17}	1.4393×10^{20}
1 erg	=	10^{-7}	1	6.2415×10^{11}	5.0340×10^{15}	2.2937×10^{10}	1.4393×10^{13}
1 eV	=	1.6022×10^{-19}	1.6022×10^{-12}	1	8065.5	3.6749×10^{-2}	23.060
1 cm^{-1}	=	1.9865×10^{-23}	1.9865×10^{-16}	$1.2398_5 \times 10^{-4}$	1	4.55634×10^{-6}	2.8591×10^{-3}
1 hartree	=	4.3598×10^{-18}	4.3598×10^{-11}	27.212	219474.6	1	627.51
1 kcal/mole	=	6.9478×10^{-21}	6.9478×10^{-14}	4.3364×10^{-2}	349.75	1.5936×10^{-3}	1

NOTE: The relationships involving cm^{-1} or kcal/mole are correspondences rather than actual equalities. The wave number $1/\lambda$ (commonly expressed in cm^{-1}) is often used as a measure of energy; a photon with energy E of 1.99×10^{-23} J has a wave number $1/\lambda = E/hc$ of 1 cm^{-1}. If a mole of molecules has an energy of 1 kcal, then each molecule has an energy of 0.043 eV.

อักษรกรีก

Alpha	Α	α
Beta	Β	β
Gamma	Γ	γ
Delta	Δ	δ
Epsilon	Ε	ϵ
Zeta	Ζ	ζ
Eta	Η	η
Theta	Θ	θ
Iota	Ι	ι
Kappa	Κ	κ
Lambda	Λ	λ
Mu	Μ	μ
Nu	Ν	ν
Xi	Ξ	ξ
Omicron	Ο	o
Pi	Π	π
Rho	Ρ	ρ
Sigma	Σ	σ
Tau	Τ	τ
Upsilon	Υ	υ
Phi	Φ	ϕ
Chi	Χ	χ
Psi	Ψ	ψ
Omega	Ω	ω

ภาคผนวกที่ 3

หน่วยเอสไอมูลฐานพร้อมชื่อและสัญลักษณ์

Physical quantity	*Name of unit*	*Symbol*
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Luminous intensity	candela	cd
Amount of substance	mole	mol

หน่วยเอสไออนุพันธ์บางหน่วยพร้อมชื่อและสัญลักษณ์

Physical quantity	*Name of SI unit*	*Symbol for SI unit*	*Definition of SI unit*
Energy	joule	J	$kg \cdot m^2 \cdot s^{-2}$
Force	newton	N	$kg \cdot m \cdot s^{-2}$
Power	watt	W	$kg \cdot m^2 \cdot s^{-3}$ (= $J \cdot s^{-1}$)
Electric charge	coulomb	C	$A \cdot s$
Electric potential	volt	V	$kg \cdot m^2 \cdot s^{-3} \cdot A^{-1}$ (= $J \cdot A^{-1} \cdot s^{-1}$)
Magnetic flux density	tesla	T	$kg \cdot s^{-2} \cdot A^{-1}$ (= $V \cdot s \cdot m^{-2}$)
Frequency	hertz	Hz	s^{-1} (cycle per second)

หน่วยที่ไม่ใช่หน่วยเอสไอบางหน่วยที่ยังคงนิยมใช้อยู่

Unit	*Symbol*	*SI value*
Angstrom	Å	10^{-10} m
Micron	μ	10^{-6} m
Calorie	cal	4.184 J
Gauss	G	10^{-4} T
Debye	D	3.3356×10^{-30} C m

หน่วยอะตอมิกที่เทียบไปเป็นหน่วยเอสไอ

Quantity	*Atomic unit*	*SI equivalent*
Mass	$m = 1$ (electron mass)	9.1091×10^{-31} kg
Charge	$\|e\| = 1$ (electron charge)	1.6021×10^{-19} C
Angular momentum	$\hbar = 1$	1.0545×10^{-34} J·s
Permittivity	$\kappa_0 = 4\pi\epsilon_0 = 1$	1.1126×10^{-10} $C^2 \cdot J^{-1} \cdot m^{-1}$
Length	$\kappa_0\hbar^2/me^2 = a_0 = 1$ (bohr) (Bohr radius)	5.29177×10^{-11} m
Energy	$me^4/\kappa_0^2\hbar^2 = e^2/\kappa_0 a_0 = 1$ (hartree) (twice the ionization energy of atomic hydrogen)	4.35944×10^{-18} J
Time	$\kappa_0^2\hbar^3/me^4 = 1$ (period of an electron in the first Bohr orbit)	2.41889×10^{-17} s
Speed	$e^2/\kappa_0\hbar = 1$ (speed of an electron in the first Bohr orbit)	2.18764×10^{6} $m \cdot s^{-1}$
Electric potential	$me^3/\kappa_0^2\hbar^2 = e/\kappa_0 a_0 = 1$ (potential energy of an electron in the first Bohr orbit)	27.211 V
Magnetic dipole moment	$e\hbar/m = 1$ (twice a Bohr Magneton)	1.85464×10^{-23} $J \cdot T^{-1}$

การเรียกชื่อ "อุปสรรค" นำหน้าหน่วย

Fraction	*Prefix*	*Symbol*	*Prefix*	*Multiple*	*Symbol*
10^{-1}	deci	d	deka	10	da
10^{-2}	centi	c	hecto	10^{2}	h
10^{-3}	milli	m	kilo	10^{3}	k
10^{-6}	micro	μ	mega	10^{6}	M
10^{-9}	nano	n	giga	10^{9}	G
10^{-12}	pico	p	tera	10^{12}	T
10^{-15}	femto	f	peta	10^{15}	P
10^{-18}	atto	a	exa	10^{18}	E

ภาคผนวกที่ 4

ตารางธาตุ

IA	IIA	IIIA	IVA	VA	VIA	VIIA	VIII			IB	IIB	IIIB	IVB	VB	VIB	VIIB	O
$_{1}$H 1.008																	$_{2}$He 4.003
$_{3}$Li 6.941	$_{4}$Be 9.012											$_{5}$B 10.81	$_{6}$C 12.01	$_{7}$N 14.01	$_{8}$O 16.00	$_{9}$F 19.00	$_{10}$Ne 20.18
$_{11}$Na 22.99	$_{12}$Mg 24.31											$_{13}$Al 26.98	$_{14}$Si 28.09	$_{15}$P 30.97	$_{16}$S 32.06	$_{17}$Cl 35.45	$_{18}$Ar 39.95
$_{19}$K 39.10	$_{20}$Ca 40.08	$_{21}$Sc 44.96	$_{22}$Ti 47.90	$_{23}$V 50.94	$_{24}$Cr 52.00	$_{25}$Mn 54.94	$_{26}$Fe 55.85	$_{27}$Co 58.93	$_{28}$Ni 58.71	$_{29}$Cu 63.55	$_{30}$Zn 65.37	$_{31}$Ga 69.72	$_{32}$Ge 72.50	$_{33}$As 74.92	$_{34}$Se 78.96	$_{35}$Br 79.90	$_{36}$Kr 83.80
$_{37}$Rb 85.47	$_{38}$Sr 87.62	$_{39}$Y 88.91	$_{40}$Zr 91.22	$_{41}$Nb 92.91	$_{42}$Mo 95.94	$_{43}$Tc 98.91	$_{44}$Ru 101.1	$_{45}$Rh 102.9	$_{46}$Pd 106.4	$_{47}$Ag 107.9	$_{48}$Cd 112.4	$_{49}$In 114.8	$_{50}$Sn 118.7	$_{51}$Sb 121.8	$_{52}$Te 127.6	$_{53}$I 126.9	$_{54}$Xe 131.1
$_{55}$Cs 132.9	$_{56}$Ba 137.3	$_{57}$La 138.9	$_{72}$Hf 178.5	$_{73}$Ta 180.9	$_{74}$W 183.9	$_{75}$Re 186.2	$_{76}$Os 190.2	$_{77}$Ir 192.2	$_{78}$Pt 195.1	$_{79}$Au 197.0	$_{80}$Hg 200.6	$_{81}$Ti 204.4	$_{82}$Pb 207.2	$_{83}$Bi 209.0	$_{84}$Po (210)	$_{85}$At (210)	$_{86}$Rn (222)
$_{87}$Fr (223)	$_{88}$Ra 226.0	$_{89}$Ac (227)															

Lanthanides	$_{57}$La 138.9	$_{58}$Ce 140.1	$_{59}$Pr 140.9	$_{60}$Nd 144.2	$_{61}$Pm (147)	$_{62}$Sm 150.4	$_{63}$Eu 152.0	$_{64}$Gd 157.3	$_{65}$Tb 158.9	$_{66}$Dy 162.5	$_{67}$Ho 164.9	$_{68}$Er 167.3	$_{69}$Tm 168.9	$_{70}$Yb 173.0	$_{71}$Lu 175.0
Actinides	$_{89}$Ac (227)	$_{90}$Th 232.0	$_{91}$Pa 231.0	$_{92}$U 238.0	$_{93}$Np 237.0	$_{94}$Pu (242)	$_{95}$Am (243)	$_{96}$Cm (248)	$_{97}$Bk (247)	$_{98}$Cf (251)	$_{99}$Es (254)	$_{100}$Fm (253)	$_{101}$Md (256)	$_{102}$No (254)	103Lw (257)

ภาคผนวกที่ 5

การจัดอิเล็กตรอนในอะตอม

Z	Element	Electron Configuration	Z	Element	Electron Configuration	Z	Element	Electron Configuration
1	H	$1s$	37	Rb	$[Kr]5s$	71	Lu	$[Xe]4f^{14}5d6s^2$
2	He	$1s^2$	38	Sr	$[Kr]5s^2$	72	Hf	$[Xe]4f^{14}5d^26s^2$
3	Li	$[He]2s$	39	Y	$[Kr]4d5s^2$	73	Ta	$[Xe]4f^{14}5d^36s^2$
4	Be	$[He]2s^2$	40	Zr	$[Kr]4d^25s^2$	74	W	$[Xe]4f^{14}5d^46s^2$
5	B	$[He]2s^22p$	41	Nb	$[Kr]4d^45s$	75	Re	$[Xe]4f^{14}5d^56s^2$
6	C	$[He]2s^22p^2$	42	Mo	$[Kr]4d^55s$	76	Os	$[Xe]4f^{14}5d^66s^2$
7	N	$[He]2s^22p^3$	43	Tc	$[Kr]4d^55s^2$	77	Ir	$[Xe]4f^{14}5d^76s^2$
8	O	$[He]2s^22p^4$	44	Ru	$[Kr]4d^75s$	78	Pt	$[Xe]4f^{14}5d^96s$
9	F	$[He]2s^22p^5$	45	Rh	$[Kr]4d^85s$	79	Au	$[Xe]4f^{14}5d^{10}6s$
10	Ne	$[He]2s^22p^6$	46	Pd	$[Kr]4d^{10}$	80	Hg	$[Xe]4f^{14}5d^{10}6s^2$
11	Na	$[Ne]3s$	47	Ag	$[Kr]4d^{10}5s$	81	Tl	$[Xe]4f^{14}5d^{10}6s^26p$
12	Mg	$[Ne]3s^2$	48	Cd	$[Kr]4d^{10}5s^2$	82	Pb	$[Xe]4f^{14}5d^{10}6s^26p^2$
13	Al	$[Ne]3s^23p$	49	In	$[Kr]4d^{10}5s^25p$	83	Bi	$[Xe]4f^{14}5d^{10}6s^26p^3$
14	Si	$[Ne]3s^23p^2$	50	Sn	$[Kr]4d^{10}5s^25p^2$	84	Po	$[Xe]4f^{14}5d^{10}6s^26p^4$
15	P	$[Ne]3s^23p^3$	51	Sb	$[Kr]4d^{10}5s^25p^3$	85	At	$[Xe]4f^{14}5d^{10}6s^26p^5$
16	S	$[Ne]3s^23p^4$	52	Te	$[Kr]4d^{10}5s^25p^4$	86	Rn	$[Xe]4f^{14}5d^{10}6s^26p^6$
17	Cl	$[Ne]3s^23p^5$	53	I	$[Kr]4d^{10}5s^25p^5$	87	Fr	$[Rn]7s$
18	Ar	$[Ne]3s^23p^6$	54	Xe	$[Kr]4d^{10}5s^25p^6$	88	Ra	$[Rn]7s^2$
19	K	$[Ar]4s$	55	Cs	$[Xe]6s$	89	Ac	$[Rn]6d7s^2$
20	Ca	$[Ar]4s^2$	56	Ba	$[Xe]6s^2$	90	Th	$[Rn]6d^27s^2$
21	Sc	$[Ar]3d4s^2$	57	La	$[Xe]5d6s^2$	91	Pa	$[Rn]5f^26d7s^2$
22	Ti	$[Ar]3d^24s^2$	58	Ce	$[Xe]4f5d6s^2$	92	U	$[Rn]5f^36d7s^2$
23	V	$[Ar]3d^34s^2$	59	Pr	$[Xe]4f^36s^2$	93	Np	$[Rn]5f^46d7s^2$
24	Cr	$[Ar]3d^54s$	60	Nd	$[Xe]4f^46s^2$	94	Pu	$[Rn]5f^67s^2$
25	Mn	$[Ar]3d^54s^2$	61	Pm	$[Xe]4f^56s^2$	95	Am	$[Rn]5f^77s^2$
26	Fe	$[Ar]3d^64s^2$	62	Sm	$[Xe]4f^66s^2$	96	Cm	$[Rn]5f^76d7s^2$
27	Co	$[Ar]3d^74s^2$	63	Eu	$[Xe]4f^76s^2$	97	Bk	$[Rn]5f^97s^2$
28	Ni	$[Ar]3d^84s^2$	64	Gd	$[Xe]4f^75d6s^2$	98	Cf	$[Rn]5f^{10}7s^2$
29	Cu	$[Ar]3d^{10}4s$	65	Tb	$[Xc]4f^96s^2$	99	Es	$[Rn]5f^{11}7s^2$
30	Zn	$[Ar]3d^{10}4s^2$	66	Dy	$[Xe]4f^{10}6s^2$	100	Fm	$[Rn]5f^{12}7s^2$
31	Ga	$[Ar]3d^{10}4s^24p$	67	Ho	$[Xe]4f^{11}6s^2$	101	Md	$[Rn]5f^{13}7s^2$
32	Ge	$[Ar]3d^{10}4s^24p^2$	68	Er	$[Xe]4f^{12}6s^2$	102	No	$[Rn]5f^{14}7s^2$
33	As	$[Ar]3d^{10}4s^24p^3$	69	Tm	$[Xe]4f^{13}6s^2$	103	Lr	$[Rn]5f^{14}6d7s^2$
34	Se	$[Ar]3d^{10}4s^24p^4$	70	Yb	$[Xe]4f^{14}6s^2$			
35	Br	$[Ar]3d^{10}4s^24p^5$						
36	Kr	$[Ar]3d^{10}4s^24p^6$						

ภาคผนวกที่ 6

สูตรคณิตศาสตร์บางสูตรที่ต้องใช้

$$\sin \alpha \sin \beta = \frac{1}{2} \cos (\alpha - \beta) - \frac{1}{2} \cos (\alpha + \beta)$$

$$\cos \alpha \cos \beta = \frac{1}{2} \cos (\alpha - \beta) + \frac{1}{2} \cos (\alpha + \beta)$$

$$\sin \alpha \cos \beta = \frac{1}{2} \sin (\alpha + \beta) + \frac{1}{2} \sin (\alpha - \beta)$$

$$\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos (\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$e^{\pm i\theta} = \cos \theta \pm i \sin \theta$$

$$\cos \theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2 + \frac{1}{3!} f''(a)(x - a)^3 + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \ldots \qquad x^2 < 1$$

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!} x^2 \pm \frac{n(n-1)(n-1)}{3!} x^3 + \ldots \qquad x^2 < 1$$

ภาคผนวกที่ 7
ค่าอินทิกรัลที่มีประโยชน์

$$\int x\sin bx\,dx = \frac{1}{b^2}\sin bx - \frac{x}{b}\cos bx$$

$$\int xe^{bx}\,dx = \frac{e^{bx}}{b^2}(bx - 1)$$

$$\int x^2e^{bx}\,dx = e^{bx}\left(\frac{x^2}{b} - \frac{2x}{b^2} + \frac{2}{b^3}\right)$$

$$\int_0^\infty x^n e^{-qx}\,dx = \frac{n!}{q^{n+1}} \qquad n > -1,\ q > 0$$

$$\int_t^\infty z^n e^{-az}\,dz = \frac{n!}{a^{n+1}}e^{-at}\left(1 + at + \frac{a^2t^2}{2!} + \ldots + \frac{a^nt^n}{n!}\right),\ n = 0, 1, 2, \ldots$$

$$\int_0^\infty e^{-ax^2} = \left(\frac{\pi}{4a}\right)^{1/2}$$

$$\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{1\ 3.5.....(2n-1)}{2^{n+1}\ a^n}\left(\frac{\pi}{a}\right)^{1/2} \quad \text{(n positive integer)}$$

$$\int_0^\infty x^{2n+1}e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}} \qquad \text{(n positive integer)}$$

$$\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\,dx = \int_0^a \cos\frac{n\pi x}{a}\cos\frac{m\pi x}{a}\,dx = \frac{a}{2}\,\delta_{nm}$$

ภาคผนวกที่ 8

MATHEMATICAL APPENDIX

1. The Hermite polynomials

The differential equation satisfied by the Hermite polynomials is

$$\frac{d^2H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2n\,H_n = 0. \qquad (A.1)$$

The coefficients of this equation have no singularities for finite values of x, so the solution can be expanded in a power series about any point and will be valid everywhere. Consider the expansion about the origin and let

$$H_n(x) = \sum_{s=0}^{\infty} b_s x^s. \qquad (A.2)$$

Substituting in the differential equation and equating to zero the coefficient of each power gives

$$(s+2)(s+1)b_{s+2} = (2s-2n)b_s. \qquad (A.3)$$

From the recursion formula it can be seen that two independent series can be obtained by starting with $s = 0$ and with $s = 1$, and that each series will contain only even or only odd powers of x. Furthermore $b_{n+2} = 0$ so that the solution will be a polynomial in case the terms used have the same parity as n. Only this case is of interest for the problem of the harmonic oscillator.

A few of these polynomials are as follows:

$H_0(x) = 1$ $\qquad H_1(x) = 2x$

$H_2(x) = 4x^2-2$ $\qquad H_3(x) = 8x^3-12x$

$H_4(x) = 16x^4-48x^2+12$ $\qquad H_5(x) = 32x^5-160x^3+120x$

$H_6(x) = 64x^6-480x^4+720x^2-120$

$H_7(x) = 128x^7-1344x^5+3360x^3-1680x.$

A useful form for the polynomials is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n(e^{-x^2})}{dx^n}. \tag{A.4}$$

The polynomials written out above come directly from this form, which accounts for the rather large numerical coefficients. The form (A.4) can be shown by substitution to satisfy the differential equation (A.1).

A useful recursion formula which can be derived from equation (A.4) is

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{A.5}$$

Also

$$\frac{dH_n}{dx} = 2xH_n - H_{n+1} = 2nH_{n-1}. \tag{A.6}$$

The Hermite polynomials are not orthogonal as they stand. As do all polynomials they become infinite for infinite values of the argument. However they do satisfy the relationship

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n\, n!\, \pi^{\frac{1}{2}} \delta(n, m). \tag{A.7}$$

2. The associated Legendre polynomials

For real values of the argument and for integral values of l and m ($l \geq m$), the associated Legendre polynomials may be defined by

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{\frac{1}{2}m} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l. \tag{A.8}$$

These functions satisfy the differential equation

$$(1-x^2)\frac{d^2 P_l^m}{dx^2} - 2x\frac{dP_l^m}{dx} + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right] P_l^m = 0 \tag{A.9}$$

and are defined for both $\pm m$. They also satisfy the integral relationship

$$\int_{-1}^{1} P_l^m(x)\, P_{l'}^m(x)\, dx = \frac{2(l+m)!}{(2l+1)(l-m)!} \delta(l, l'). \tag{A.10}$$

A few of these functions are as follows:

$P_0^0 = 1$	$P_2^2 = 3(1-x^2)$
$P_1^1 = -(1-x^2)^{\frac{1}{2}}$	$P_2^1 = -3(1-x^2)^{\frac{1}{2}}\, x$
$P_1^0 = x$	$P_2^0 = \frac{1}{2}(3x^2-1)$
$P_1^{-1} = \frac{1}{2}(1-x^2)^{\frac{1}{2}}$	$P_2^{-1} = \frac{1}{2}(1-x^2)^{\frac{1}{2}}\, x$
	$P_2^{-2} = \frac{1}{8}(1-x^2)$
$P_3^3 = -15(1-x^2)^{\frac{3}{2}}$	$P_4^4 = 105(1-x^2)^2$
	$P_4^3 = -105(1-x^2)^{\frac{3}{2}}\, x$
$P_3^2 = 15(1-x^2)\, x$	$P_4^2 = \frac{15}{2}(1-x^2)(7x^2-1)$
$P_3^1 = -\frac{3}{2}(1-x^2)^{\frac{1}{2}}(5x^2-1)$	$P_4^1 = -\frac{5}{2}(1-x^2)^{\frac{1}{2}}(7x^3-3x)$
$P_3^0 = \frac{1}{2}(5x^2-3)\, x$	$P_4^0 = \frac{1}{8}(35x^4-30x^2+3)$
$P_3^{-1} = \frac{1}{8}(1-x^2)^{\frac{1}{2}}(5x^2-1)$	$P_4^{-1} = \frac{1}{8}(1-x^2)^{\frac{1}{2}}(7x^2-3)\, x$
$P_3^{-2} = \frac{1}{8}(1-x^2)\, x$	$P_4^{-2} = \frac{1}{48}(1-x^2)(7x^2-1)$
$P_3^{-3} = \frac{1}{48}(1-x^2)^{\frac{3}{2}}$	$P_4^{-3} = \frac{1}{48}(1-x^2)^{\frac{3}{2}}\, x$
	$P_4^{-4} = \frac{1}{384}(1-x^2)^2$

The numerical coefficients of the above examples, and the signs, follow from the form (A.8); they are not required by the differential equation.

The integral relationship (A.10) provides a means of normalizing these functions between 0 and 1 which is useful in series of orthogonal functions. One may define

$$\Pi_l^m(x) = \left[\frac{(2l+1)(l-m)!}{2(l+m)!}\right]^{\frac{1}{2}} P_l^m . \tag{A.11}$$

Then

$$\int_{-1}^{1} \Pi_l^m \, \Pi_{l'}^m \, \mathrm{d}x = \delta(l, l') .$$

3. Surface spherical harmonics

The associated Legendre polynomials can be used to form a complete set of surface spherical harmonics. These normalized and orthogonal functions may be defined as

$$Y(\vartheta, \varphi) = \Pi_l^m(\cos\vartheta)\, \mathrm{e}^{\mathrm{i}m\varphi}/(2\pi)^{\frac{1}{2}} . \tag{A.12}$$

Using the above definitions these functions may be formed with both positive and negative values of m in a satisfyingly symmetric way. They satisfy the differential equation, which can be obtained from equation (A.9),

$$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial Y_l^m}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\,\frac{\partial^2 Y_l^m}{\partial\varphi^2}=l(l+1)Y_l^m. \qquad (A.13)$$

A few of these normalized functions are as follows:

$$\Pi_0^0 = (\tfrac{1}{2})^{\frac{1}{2}}$$

$$\Pi_1^1 = -(\tfrac{3}{4})^{\frac{1}{2}}\sin\vartheta$$
$$\Pi_1^0 = (\tfrac{3}{2})^{\frac{1}{2}}\cos\vartheta$$
$$\Pi_1^{-1} = (\tfrac{3}{4})^{\frac{1}{2}}\sin\vartheta$$

$$\Pi_3^{-3} = -(\tfrac{35}{32})^{\frac{1}{2}}\sin^3\vartheta$$
$$\Pi_3^2 = (\tfrac{105}{16})^{\frac{1}{2}}\sin^2\vartheta\cos\vartheta$$
$$\Pi_3^1 = -(\tfrac{21}{32})^{\frac{1}{2}}\sin\vartheta\,(5\cos^2\vartheta-1)$$
$$\Pi_3^0 = (\tfrac{7}{8})^{\frac{1}{2}}(5\cos^3\vartheta-3\cos\vartheta)$$
$$\Pi_3^{-1} = (\tfrac{21}{32})^{\frac{1}{2}}\sin\vartheta\,(5\cos^2\vartheta-1)$$
$$\Pi_3^{-2} = (\tfrac{105}{16})^{\frac{1}{2}}\sin^2\vartheta\cos\vartheta$$
$$\Pi_3^{-3} = (\tfrac{35}{32})^{\frac{1}{2}}\sin^3\vartheta$$

$$\Pi_2^2 = (\tfrac{15}{16})^{\frac{1}{2}}\sin^2\vartheta$$
$$\Pi_2^1 = -(\tfrac{15}{4})^{\frac{1}{2}}\sin\vartheta\cos\vartheta$$
$$\Pi_2^0 = (\tfrac{5}{8})^{\frac{1}{2}}(3\cos^2\vartheta-1)$$
$$\Pi_2^{-1} = (\tfrac{15}{4})^{\frac{1}{2}}\sin\vartheta\cos\vartheta$$
$$\Pi_2^{-2} = (\tfrac{15}{16})^{\frac{1}{2}}\sin^2\vartheta$$

$$\Pi_4^4 = (\tfrac{315}{256})^{\frac{1}{2}}\sin^4\vartheta$$
$$\Pi_4^3 = -(\tfrac{315}{32})^{\frac{1}{2}}\sin^3\vartheta\cos\vartheta$$
$$\Pi_4^2 = (\tfrac{45}{64})^{\frac{1}{2}}\sin^2\vartheta\cos\vartheta\,(7\cos^2\vartheta-1)$$
$$\Pi_4^1 = -(\tfrac{45}{32})^{\frac{1}{2}}\sin\vartheta\,(7\cos^3\vartheta-3\cos\vartheta)$$
$$\Pi_4^0 = (\tfrac{9}{128})^{\frac{1}{2}}(35\cos^4\vartheta-30\cos^2\vartheta+3)$$
$$\Pi_4^{-1} = (\tfrac{45}{32})^{\frac{1}{2}}\sin\vartheta\,(7\cos^3\vartheta-3\cos\vartheta)$$
$$\Pi_4^{-2} = (\tfrac{45}{64})^{\frac{1}{2}}\sin^2\vartheta\,(7\cos^2\vartheta-1)$$
$$\Pi_4^{-3} = (\tfrac{315}{32})^{\frac{1}{2}}\sin^3\vartheta\cos\vartheta$$
$$\Pi_4^{-4} = (\tfrac{315}{256})^{\frac{1}{2}}\sin^4\vartheta.$$

There are numerous relationships among these functions which are often useful. Among them are

$$\cos\vartheta\,\Pi_l^m = \left[\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}}\Pi_{l+1}^m + \left[\frac{(l+m)(l-m)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}}\Pi_{l-1}^m \qquad (A.14)$$

$$\cos\vartheta\,\Pi_l^m = \left[\frac{(2l+1)(l-m+1)}{(2l+3)(l+m+1)}\right]^{\frac{1}{2}}\Pi_{l+1}^m - \left[\frac{l-m}{l+m+1}\right]^{\frac{1}{2}}\sin\vartheta\,\Pi_l^{m+1} \qquad (A.15)$$

$$\sin\vartheta\,\Pi_l^m = -\left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)}\right]^{\frac{1}{2}}\Pi_{l+1}^{m+1} + \left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}}\Pi_{l-1}^{m+1} \qquad (A.16)$$

$$\sin\vartheta\,\Pi_l^m = \left[\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}} \Pi_{l+1}^{m-1} - \left[\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}\right]^{\frac{1}{2}} \Pi_{l-1}^{m-1} \tag{A.17}$$

$$\sin\vartheta\,\frac{\mathrm{d}\,\Pi_l^m}{\mathrm{d}\,\vartheta} = l\cos\vartheta\,\Pi_l^m - \left(\frac{2l+1}{2l-1}\right)^{\frac{1}{2}} (l^2-m^2)^{\frac{1}{2}}\,\Pi_{l-1}^m \tag{A.18}$$

$$\sin\vartheta\,\frac{\mathrm{d}\,\Pi_l^m}{\mathrm{d}\,\vartheta} = -(l+1)\cos\vartheta\,\Pi_l^m + \left(\frac{2l+1}{2l+3}\right)^{\frac{1}{2}} [(l+m+1)(l-m+1)]^{\frac{1}{2}}\,\Pi_{l+1}^m . \tag{A.19}$$

In addition to equation (A.11) the functions $Y_l^m(\vartheta,\varphi)$ also satisfy the following relationships:

$$(l_x+\mathrm{i}l_y)Y_l^m = \frac{\hbar}{\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\varphi}\left(\mathrm{i}\frac{\partial}{\partial\vartheta} - \cot\vartheta\,\frac{\partial}{\partial\varphi}\right) Y_l^m$$

$$= \hbar[(l-m)(l+m+1)]^{\frac{1}{2}}\,Y_l^{m+1}, \tag{A.20}$$

$$(l_x-\mathrm{i}l_y)Y_l^m = -\frac{\hbar}{\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}\varphi}\left(\mathrm{i}\frac{\partial}{\partial\vartheta} + \cot\vartheta\,\frac{\partial}{\partial\varphi}\right) Y_l^m$$

$$= \hbar[(l+m)(l-m+1)]^{\frac{1}{2}}\,Y_l^{m-1}. \tag{A.21}$$

Another important relationship between spherical harmonics is the addition theorem. If (ϑ, φ) and (ϑ', φ') are the coordinates of two directions and θ is the angle between them

$$\cos\theta = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi-\varphi') \tag{A.22}$$

and

$$\{\tfrac{1}{2}(2l+1)\}^{\frac{1}{2}}\,P_l(\cos\theta) = \sum_{m=-l}^{l} \Pi_l^m(\cos\vartheta')\,\Pi_l^m(\cos\vartheta)\,\mathrm{e}^{\mathrm{i}m(\varphi-\varphi')}. \tag{A.23}$$

4. The Laguerre polynomials

The Laguerre polynomials may be defined as

$$L_k(x) = \mathrm{e}^x\,\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^k\,\mathrm{e}^{-x}). \tag{A.24}$$

This satisfies the differential equation

$$x\,\frac{\mathrm{d}^2L_k}{\mathrm{d}x^2} + (1-x)\,\frac{\mathrm{d}L_k}{\mathrm{d}x} + kL_k = 0. \tag{A.25}$$

Further differentiation of this equation leads to

$$x\frac{d^2L_k^s}{dx^2} + (s+1-x)\frac{dL_k^s}{dx} + (k-s)L_k^s = 0 \qquad (A.26)$$

where

$$L_k^s(x) = \frac{d^s}{dx^s}L_k(x) \qquad (A.27)$$

and is called the associated Laguerre polynomial. Equation (A.26) has the form of equation (4.17).

A few of these polynomials are as follows:

$L_0 = 1$

$L_1 = 1-x \qquad L_1^1 = -1$

$L_2 = 2-4x+x^2 \qquad L_2^1 = -4+2x \qquad L_2^2 = 2$

$L_3 = 6-18x+9x^2-x^3 \quad L_3^1 = -18+18x-3x^2 \quad L_3^2 = 18-6x \quad L_3^3 = -6$

$L_4 = 24-96x+72x^2-16x^3+x^4 \quad L_4^1 = -96+144x-48x^2+4x^3$

$L_4^2 = 144-96x+12x^2 \qquad L_4^3 = -96+24x \qquad L_4^4 = 24.$

The associated Laguerre polynomials satisfy the orthogonality relationship

$$\int_0^\infty L_k^s(x)L_{k'}^s(x)\,e^{-x}x^{s+1}\,dx = \frac{(2k-s+1)\,(k!)^s}{(k-s)!}\,\delta(k,k'). \qquad (A.28)$$

5. Bessel functions

The Bessel functions satisfy the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-m^2)y = 0. \qquad (A.29)$$

Solutions of this equation, valid for all values of x, can be expressed in power series about the origin. Hence the usual definition of a Bessel function is

$$J_m(x) = \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m\left\{1 - \frac{1}{m+1}\left(\frac{x}{2}\right)^2 + \frac{1}{(m+1)(m+2)}\frac{1}{2!}\left(\frac{x}{2}\right)^4 + \ldots\right. \qquad (A.30)$$

When m is an integer the power series for $(-m)$ cannot be used and other methods must be employed to find the second solution. It will not be finite at $x = 0$. When m is not an integer, a second solution may be taken to be

$$Y_m(x) = \frac{\cos(m\pi)J_m(x) - J_{-m}(x)}{\sin m\pi}. \qquad (A.31)$$

When m is half integral, the Bessel functions take on a particularly useful form in terms of trigonometric functions,

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x \qquad J_{-\frac{1}{2}} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x$$

$$J_{\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \frac{\sin x}{x} - \cos x \qquad J_{-\frac{3}{2}} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} - \frac{\cos x}{x} - \sin x$$

$$J_{\frac{5}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left(\frac{3}{x^2} - 1\right) \sin x - \frac{3}{x} \cos x$$

$$J_{-\frac{5}{2}} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \frac{3}{x} \sin x + \left(\frac{3}{x^2} - 1\right) \cos x.$$

As might be inferred from these forms, the half integral order Bessel functions approach a trigonometric function divided by $x^{\frac{1}{2}}$ as $x \to \infty$. This behavior is general and

$$J_n(x) \to \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} [\cos\{x - \tfrac{1}{2}(n+\tfrac{1}{2})\pi\} - \frac{4n^2-1}{8x} \sin\{x - \tfrac{1}{2}(n+\tfrac{1}{2})\pi\}] \qquad (A.32)$$

as $x \to \infty$.

Among the numerous relationships between various Bessel functions are the following

$$J_m(x) = \frac{x}{2m}\{J_{m-1}(x) + J_{m+1}(x)\} \qquad (A.33)$$

$$\frac{dJ_m}{dx} = \tfrac{1}{2}\{J_{m-1}(x) - J_{m+1}(x)\} \qquad (A.34)$$

$$\frac{d}{dx}[x^l J_m(x)] = \frac{x^l}{2m}\{(l+m)J_{m-1}(x) + (l-m)J_{m+1}(x)\} \qquad (A.35)$$

$$\frac{d}{dx}[x^{-l} J_m(x)] = \frac{-x^{-l}}{2m}\{(l-m)J_{m-1}(x) + (l+m)J_{m+1}(x)\}. \qquad (A.36)$$

Since the Bessel functions approach zero only as $1/x^{\frac{1}{2}}$ as $x \to \infty$, integral relationships are useful especially when the integration is carried out between roots of the functions, not when carried to infinity. One such relationship is

$$\int_a^b J_m(kx)\,J_m(lx)\,x\,\mathrm{d}x = \frac{1}{k^2-l^2}\{lx\,J_m(kx)\,J'_m(lx) - kx\,J'_m(kx)\,J_m(lx)\}\,|_a^b. \tag{A.37}$$

If the limits of integration ($x = a$, $x = b$) are such that the right-hand side vanishes,

$$\int_a^b J_m(kx)\,J_m(lx)\,x\,\mathrm{d}x = 0, \quad l \neq k. \tag{A.38}$$

If $k = l$

$$\int [J_m(kx)]^2 x\,\mathrm{d}x = \left\{\tfrac{1}{2}\left(x^2 - \frac{m^2}{k^2}\right)[J_m(kx)]^2 + \tfrac{1}{2}x^2[J'_m(kx)]^2\right\}. \tag{A.39}$$

6. Spherical Bessel functions

The Bessel functions described above frequently appear in problems of cylindrical symmetry. To treat problems of spherical symmetry Morse introduced the spherical Bessel functions defined by

$$j_n(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(x). \tag{A.40}$$

These satisfy the differential equation

$$\frac{\mathrm{d}^2 j_n(x)}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}\,j_n(x)}{\mathrm{d}x} + \left[1 - \frac{n(n+1)}{x^2}\right] j_n(x) = 0. \tag{A.41}$$

A few of these functions with $n \geq 0$ are

$$j_0(x) = \frac{1}{x}\sin x \qquad j_1(x) = \frac{1}{x}\left(\frac{\sin x}{x} - \cos x\right)$$

$$j_2(x) = \frac{1}{x}\left\{\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right\}$$

$$j_3(x) = \frac{1}{x}\left\{\left(\frac{15}{x^3} - \frac{6}{x}\right)\sin x - \left(\frac{15}{x^2} - 1\right)\cos x\right\}.$$

There are various relationships which follow from the corresponding relationships between the standard Bessel functions. Among them are

$$j_m(x) = \frac{x}{2m+1}\{j_{m-1}(x)+j_{m+1}(x)\} \qquad \text{(A.42)}$$

$$\frac{\mathrm{d}j_m(x)}{\mathrm{d}x} = \frac{1}{2m+1}\{m j_{m-1}(x)-(m+1)j_{m+1}(x)\} \qquad \text{(A.43)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^l j_m(x)] = \frac{x^l}{2m+1}\{(l+m)j_{m-1}(x)+(l-m-1)j_{m+1}(x)\} \qquad \text{(A.44)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-l} j_m(x)] = -\frac{x^{-l}}{2m+1}\{(l-m)j_{m-1}(x)+(l+m+1)j_{m+1}(x)\} \qquad \text{(A.45)}$$

$$\int j_m(kx)j_m(lx)x^2\,\mathrm{d}x = \frac{x^{\frac{1}{2}}}{k^2-l^2}\{j_m(kx)\frac{\mathrm{d}}{\mathrm{d}x}[x^{\frac{1}{2}}j_m(x)]-j_m(lx)\frac{\mathrm{d}}{\mathrm{d}x}[x^{\frac{1}{2}}j_m(kx)]\} \qquad \text{(A.46)}$$

$$\int [j_m(kx)]^2x^2\,\mathrm{d}x = \left(\tfrac{1}{2}x^3-\frac{(m+\frac{1}{2})^2x}{2k^2}\right)[j_m(kx)]^2+\frac{x^2}{2k^2}\left\{\frac{\mathrm{d}}{\mathrm{d}x}[x^{1}j_m(kx)]\right\}^2. \qquad \text{(A.47)}$$
