ภาคผนวก 2
สูตรอินทีกรัล

Some Elementary Forms

1. $\int df(x) = \int f'(x)\,dx = f(x) + C.$

2. $\int a\,du = a\int du.$

3. $\int (du \pm dv + dw \pm \ldots) = \int du + \int dv + \int dw \pm \ldots$

4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C.$ $\qquad (n \neq -1)$

5. $\int \dfrac{du}{u} = \ln u + C.$

Rational Forms containing $a + bu$

See also the Binomial Reduction Formulas 96-104.

6. $\int (a+bu)^n\,du = \dfrac{(a+bu)^{n+1}}{b(n+1)} + C.$ $\qquad (n \neq -1)$

7. $\int \dfrac{du}{a+bu} = \dfrac{1}{b}\ln(a+bu) + C.$

8. $\int \dfrac{u\,du}{a+bu} = \dfrac{1}{b^2}[a + bu - a\ln(a+bu)] + C.$

9. $\int \dfrac{u^2\,du}{a+bu} = \dfrac{1}{b^3}\left[\dfrac{1}{2}(a+bu)^2 - 2a(a+bu) + a^2\ln(a+bu)\right] + C.$

10. $\int \dfrac{u\,du}{(a+bu)^2} = \dfrac{1}{b^2}\left[\dfrac{a}{a+bu} + \ln(a+bu)\right] + C.$

11. $\int \dfrac{u^2\,du}{(a+bu)^2} = \dfrac{1}{b^3}\left[a + bu - \dfrac{a^2}{a+bu} - 2a\ln(a+bu)\right] + C.$

12. $\int \dfrac{u\,du}{(a+bu)^3} = \dfrac{1}{b^2}\left[-\dfrac{1}{a+bu} + \dfrac{a}{2(a+bu)^2}\right] + C.$

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13. $$\int \frac{du}{u(a + bu)} = -\frac{1}{a} \ln \left(\frac{a + bu}{u} \right) + C.$$

14. $$\int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2} \ln \left(\frac{a + bu}{u} \right) + C.$$

15. $$\int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} - \frac{1}{a^2} \ln \left(\frac{a + bu}{u} \right) + C.$$

Rational Forms containing $a^2 \pm b^2u^2$

16. $$\int \frac{du}{a^2 + b^2u^2} = \frac{1}{ab} \arctan \frac{bu}{a} + C.$$

17. $$\int \frac{du}{a^2 - b^2u^2} = \frac{1}{2ab} \ln \left(\frac{a + bu}{a - bu} \right) + C. \qquad (a^2 > b^2u^2)$$

$$\int \frac{du}{b^2u^2 - a^2} = \frac{1}{2ab} \ln \left(\frac{bu - a}{(bu + a)} \right) + C. \qquad (a^2 < b^2u^2)$$

18. $$\int u(a^2 \pm b^2u^2)^n du = \frac{(a^2 \pm b^2u^2)^{n+1}}{\pm 2b^2(n + 1)} + C. \qquad (n \neq -1)$$

19. $$\int \frac{u\,du}{a^2 \pm b^2u^2} = \frac{1}{\pm 2b^2} \ln (a^2 \pm b^2u^2) + C.$$

20. $$\int \frac{u^m\,du}{(a^2 \pm b^2u^2)^p} = \frac{u^{m-1}}{\pm b^2(m - 2p + 1)(a^2 \pm b^2u^2)^{p-1}} - \frac{a^2(m - 1)}{\pm b^2(m - 2p + 1)} \int \frac{u^{m-2}\,du}{(a^2 \pm b^2u^2)^p}$$

21. $$\int \frac{u^m\,du}{(a^2 \pm b^2u^2)^p} = \frac{u^{m+1}}{2a^2(p - 1)(a^2 \pm b^2u^2)^{p-1}} - \frac{m - 2p + 3}{2a^2(p - 1)} \int \frac{u^m\,du}{(a^2 \pm b^2u^2)^{p-1}}$$

22. $$\int \frac{du}{u(a^2 + b^2u^2)} = \frac{1}{2a^2} \ln \left(\frac{u^2}{a^2 \pm b^2u^2} \right) + C.$$

23. $$\int \frac{du}{u^m(a^2 \pm b^2u^2)^p} = -\frac{1}{a^2(m - 1)u^{m-1}(a^2 \pm b^2u^2)^{p-1}} - \frac{\pm b^2(m + 2p - 3)}{a^2(m - 1)} \int \frac{du}{u^{m-2}(a^2 \pm b^2u^2)^p}$$

24. $$\int \frac{du}{u^m(a^2 \pm b^2u^2)^p} = \frac{1}{2a^2(p-1)u^{m-1}(a^2 \pm b^2u^2)^{p-1}} + \frac{m+2p-3}{2a^2(p-1)} \int \frac{du}{u^m(a^2 \pm b^2u^2)^{p-1}}$$

Forms containing $\sqrt{a+bu}$

The intergrand may be rationalized by setting $a + bu = v^2$. See also the Binomial Reduction Formulas 96-104.

25. $$\int u\sqrt{a+bu}\,du = -\frac{2(2a-3bu)(a+bu)^{3/2}}{15b^2} + C.$$

26. $$\int u^2\sqrt{a+bu}\,du = \frac{2(8a^2-12abu+15b^2u^2)(a+bu)^{3/2}}{105b^3} + C.$$

27. $$\int u^m\sqrt{a+bu}\,du = \frac{2u^m(a+bu)^{3/2}}{b(2m+3)} - \frac{2am}{b(2m+3)} \int u^{m-1}\sqrt{a+bu}\,du.$$

28. $$\int \frac{u\,du}{\sqrt{a+bu}} = -\frac{2(2a-bu)\sqrt{a+bu}}{3b^2} + C.$$

29. $$\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2(8a^2-4abu+3b^2u^2)\sqrt{a+bu}}{15b^3} + C.$$

30. $$\int \frac{u^m\,du}{\sqrt{a+bu}} = 2\frac{u^m\sqrt{a+bu}}{b(2m+1)} - \frac{2am}{b(2m+1)} \int \frac{u^{m-1}\,du}{\sqrt{a+bu}}$$

31. $$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}} \ln\left(\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right) + C, \text{ for } a > 0.$$

32. $$\int \frac{du}{u\sqrt{a+bu}} = \frac{2}{\sqrt{-a}} \arctan\sqrt{\frac{a+bu}{-a}} + C, \text{ for } a < 0.$$

33. $$\int \frac{du}{u^m\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(m-1)u^{m-1}} - \frac{b(2m-3)}{2a(m-1)} \int \frac{du}{u^{m-1}\sqrt{a+bu}}$$

34. $$\int \frac{\sqrt{a+bu}\,du}{u} = 2\sqrt{a+bu} + a \int \frac{du}{u\sqrt{a+bu}}$$

35. $$\int \frac{\sqrt{a+bu}\,du}{u^m} = -\frac{(a+bu)^{3/2}}{a(m-1)u^{m-1}} - \frac{b(2m-5)}{2a(m-1)} \int \frac{\sqrt{a+bu}\,du}{u^{m-1}}$$

Forms containing $\sqrt{u^2 \pm a^2}$

In this group of formulas we may replace

$$\ln (u + \sqrt{u^2 + a^2}) \text{ by } \sinh^{-1}\frac{u}{a},$$

$$\ln (u + \sqrt{u^2 - a^2}) \text{ by } \cosh^{-1}\frac{u}{a},$$

$$\ln\left(\frac{a + \sqrt{u^2 + a^2}}{u}\right) \text{ by } \sinh^{-1}\frac{a}{u}.$$

36. $$\int (u^2 \pm a^2)^{1/2}\,du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln (u + \sqrt{u^2 \pm a^2}) + C.$$

37. $$\int (u^2 \pm a^2)^{n/2}\,du = \frac{u(u^2 \pm a^2)^{n/2}}{m + 1} \pm \frac{na^2}{n + 1}\int (u^2 \pm a^2)^{n/2-1}\,du. (n \neq -1)$$

38. $$\int u(u^2 \pm a^2)^{n/2}\,du = \frac{(u^2 \pm a^2)^{n/2+1}}{n + 2} + C. \qquad (n \neq -2)$$

39. $$\int u^m(u^2 \pm a^2)^{n/2}\,du = \frac{u^{m-1}(u^2 \pm a^2)^{n/2+1}}{n + m + 1} - \frac{\pm a^2(m - 1)}{n + m + 1}\int u^{m-2}(u^2 \pm a^2)^{n/2}\,du.$$

40. $$\int \frac{du}{(u^2 \pm a^2)^{1/2}} = \ln (u + \sqrt{u^2 \pm a^2}) + C.$$

41. $$\int \frac{du}{(u^2 \pm a^2)^{3/2}} = \frac{u}{+a^2\sqrt{u^2 \pm a^2}} + C.$$

42. $$\int \frac{u\,du}{(u^2 \pm a^2)^2} = \frac{(u^2 \pm a^2)^{1-n/2} + C.}{2 - n}$$

43. $$\int \frac{u^2\,du}{(u^2 \pm a^2)^{3/2}} = \frac{u}{2}\sqrt{u^2 \pm a^2} - \frac{\pm a^2}{2}\ell n (u + \sqrt{u^2 \pm a^2}) + C.$$

44. $$\int \frac{u^2\,du}{(u^2 \pm a^2)^{3/2}} = \frac{u}{\sqrt{u^2 \pm a^2}} + \ell n (u + \sqrt{u^2 \pm a^2}) + C.$$

45. $$\int \frac{u^m\,du}{(u^2 \pm a^2)^{n/2}} = \frac{u^{m-1}}{(m - n - 1)(u^2 \pm a^2)^{n-1}} - \frac{\pm a^2 (m - 1)}{m - n + 1}\int \frac{m-2\,du}{(u^2 \pm a^2)^{n/2}}$$

46. $$\frac{u^m\,du}{(u^2 \pm a^2)^2} = \frac{u^{m+1}}{\pm a^2 (n - 2)(u^2 \pm a^2)^{n/2-1}} - \frac{m - n + 3}{\pm a^2 (n - 2)}\int \frac{u^m\,du}{(u^2 \pm a^2)^{n/2-1}}$$

47. $$\int \frac{du}{u(u^2 + a^2)^{1/2}} = -\frac{1}{a}\ln\left(\frac{a + \sqrt{u^2 + a^2}}{u}\right) + C.$$

48. $$\int \frac{du}{u(u^2 - a^2)^{1/2}} = \frac{1}{a} \operatorname{arc\,sec} \frac{u}{a} + C.$$

49. $$\frac{du}{u^2(u^2 \pm a^2)^{1/2}} = -\frac{\sqrt{u^2 \pm a^2}}{\pm a^2 u} + C.$$

50. $$\int \frac{du}{u^3(u^2 + a^2)^{1/2}} = -\frac{\sqrt{u^2 + a^2}}{2a^2u^2} + \frac{1}{2a^3} \ln\left(\frac{a + \sqrt{u^2 + a^2}}{u}\right) + C.$$

51. $$\int \frac{du}{u^3(u^2 - a^2)^{1/2}} = \frac{\sqrt{u^2 - a^2}}{2a^2u^2} + \frac{1}{2a^3} \operatorname{arc\,sec} \frac{u}{a} + C.$$

52. $$\int \frac{du}{u^m(u^2 \pm a^2)^{n/2}} = -\frac{1}{\pm a^2(m - 1)u^{m-1}(u^2 \pm a^2)^{n/2 - 1}} - \frac{m + n - 3}{\pm a^2(m - 1)} \int \frac{du}{u^{m-2}(u^2 \pm a^2)^{n/2}}$$

53. $$\frac{du}{u^m(u^2 \pm a^2)^{n/2}} = \frac{1}{\pm a^2(n - 2)u^{m-1}(u^2 \pm a^2)^{n/2 - 1}} + \frac{m + n - 3}{\pm a^2(n - 2)} \int \frac{du}{u^m(u^2 \pm a^2)^{n/2 - 1}}$$

54. $$\int \frac{(u^2 + a^2)du}{u} = \sqrt{u^2 + a^2} - a \ln\left(\frac{a + \sqrt{u^2 + a^2}}{u}\right) + C.$$

55. $$\int \frac{(u^2 - a^2)^{1/2} du}{u} = \sqrt{u^2 - a^2} - a \operatorname{arc\,sec} \frac{u}{a} + C.$$

56. $$\int \frac{(u^2 \pm a^2)^{1/2} du}{u^2} = -\frac{\sqrt{u^2 \pm a^2}}{u} + \ln \sqrt{u^2 \pm a^2}) + C.$$

57. $$\int \frac{(u^2 \pm a^2)^{n/2} du}{u^m} = -\frac{(u^2 \pm a^2)^{n/2 + 1}}{\pm a^2(m - 1)u^{m-1}} - \frac{m - n - 3}{\pm a^2(m - 1)} \int \frac{(u^2 \pm a^2)^{n/2} du}{u^{m-2}}$$

58. $$\int \frac{(u^2 \pm a^2)^{n/2} du}{u^m} = \frac{(u^2 \pm a^2)^{n/2}}{(n - m + 1)u^{m-1}} + \frac{\pm a^2 n}{n - m + 1} \int \frac{(u^2 \pm a^2)^{n/2 - 1} du}{u^m}$$

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Forms containing $\sqrt{a^2 - u^2}$

59. $\int (a^2 - u^2)^{1/2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \frac{u}{a} + C.$

60. $\int (a^2 - u^2)^{n/2}\, du = \frac{u(a^2 - u^2)^{n/2}}{n + 1} + \frac{a^2 n}{n + 1} \int (a^2 - u^2)^{n/2 - 1} du. \quad (n \neq -1)$

61. $\int u^m (a^2 - u^2)^{n/2}\, du = -\frac{u\,(a^2 - u^2)^{n/2+1}}{n + 2} + C. \quad (n \neq -2)$

62. $\int u^m (a^2 - u^2)^{n/2}\, du = -\frac{u^{m-1}\,(a^2 - u^2)^{n/2+1}}{n + m + 1}$

$$+ \frac{a^2(m - 1)}{n + m + 1} \int u^{m-2}(a^2 - u^2)^{n/2}\, du.$$

63. $\int \frac{du}{(a^2 - u^2)^{1/2}} = \arcsin \frac{u}{a} + C.$

64. $\int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C.$

65. $\int \frac{u\, du}{(a^2 - u^2)^{n/2}} = \frac{(a^2 - u^2)^{1 - n/2}}{n - 2} + C.$

66. $\int \frac{u^2\, du}{(a^2 - u^2)^{1/2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \frac{u}{a} + C.$

67. $\int \frac{u^2\, du}{(a^2 - u^2)^{3/2}} = \frac{u}{\sqrt{a^2 - u^2}} - \arcsin \frac{u}{a} + C.$

68. $\int \frac{u^m\, du}{(a^2 - u^2)^{n/2}} = -\frac{u^{m-1}}{(m - n + 1)(a^2 - u^2)^{n/2 - 1}} + \frac{a^2(m - 1)}{m - n + 1} \int \frac{u^{m-2}\, du}{(a^2 - u^2)^{n/2}}$

69. $\int \frac{u^m\, du}{(a^2 - u^2)^{n/2}} = \frac{u^{m+1}}{a^2 (n - 2)(a^2 - u^2)^{n/2-1}} - \frac{m - n + 3}{a^2 (n - 2)} \int \frac{u^m\, du}{(a^2 - u^2)^{n/2-1}}$

70. $\int \frac{du}{u(a^2 - u^2)^{1/2}} = -\frac{1}{a} \ln \left(\frac{a + \sqrt{a^2 - u^2}}{u} \right) + C = -\frac{1}{a} \cosh^{-1} \frac{a}{u} + C.$

71. $\int \frac{du}{u^2(a^2 - u^2)^{1/2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u} + C.$

72. $\int \frac{du}{u^3(a^2 - u^2)^{1/2}} = -\frac{\sqrt{a^2 - u^2}}{2a^2u^2} - \frac{1}{2a^3} \ln \left(\frac{a + \sqrt{a^2 - u^2}}{u} \right) + C$

$$= -\frac{\sqrt{a^2 - u^2}}{2a^2u^2} - \frac{1}{2a^3}\cosh^{-1}\frac{a}{u} + C.$$

73. $$\int \frac{du}{u^m(a^2 - u^2)^{n/2}} = -\frac{1}{a^2(m-1)u^{m-1}(a^2 - u^2)^{n/2 - 1}} + \frac{m + n - 3}{a^2(m-1)} \int \frac{du}{u^{m-2}(a^2 - u^2)^{n/2}}$$

74. $$\int \frac{du}{u^m(a^2 - u^2)^{n/2}} = \frac{1}{a^2(n-2)u^{m-1}(a^2 - u^2)^{n/2 - 1}} + \frac{m + n - 3}{a^2(n-2)} \int \frac{du}{u^m(a^2 - u^2)^{n/2 - 1}}$$

75. $$\int \frac{(a^2 - u^2)^{1/2}\,du}{u} = \sqrt{a^2 - u^2} - a \ln\left(\frac{a + \sqrt{a^2 - u^2}}{u}\right) + C$$

$$= \sqrt{a^2 - u^2} - a\cosh^{-1}\frac{a}{u} + C.$$

76. $$\int \frac{(a^2 - u^2)^{1/2}\,du}{u^2} = -\frac{\sqrt{a^2 - u^2}}{u} - \arcsin\frac{u}{a} + C.$$

77. $$\int \frac{(a^2 - u^2)^{n/2}\,du}{u^m} = -\frac{(a^2 - u^2)^{n/2 + 1}}{a^2(m-1)u^{m-1}} + \frac{m - n - 3}{a^2(m-1)} \int \frac{(a^2 - u^2)^{n/2}\,du}{u^{m-2}}$$

78. $$\int \frac{(a^2 - u^2)^{n/2}\,du}{u^m} = \frac{(a^2 - u^2)^{n/2}}{(n - m + 1)u^{m-1}} + \frac{a^2 n}{n - m + 1} \cdot \int \frac{(a^2 - u^2)^{n/2}\,du}{u^m}$$

Forms containing $\sqrt{2au \pm u^2}$

The Binomial Reduction Formulas 96-104 may be applied by writing

$$\sqrt{2au \pm u^2} = u^{1/2}\,(2a \pm u)^{1/2}$$

79. $$\int \sqrt{2au - u^2}\,du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\arccos\left(1 - \frac{u}{a}\right) + C.$$

80. $$\int u\sqrt{2au - u^2}\,du = -\frac{3a^2 + au - 2u^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\arccos\left(1 - \frac{u}{a}\right) + C.$$

81. $$\int u^m\sqrt{2au - u^2}\,du = -\frac{u^{m-1}(2au - u^2)^{3/2}}{m + 2}$$

$$+ \frac{a(2m+1)}{m+2} \int u^{m-1}\sqrt{2au-u^2}\,du.$$

82. $$\int \frac{\sqrt{2au-u^2}\,du}{u} = \sqrt{2au-u^2} + a \arccos\left(1-\frac{u}{a}\right) + C.$$

83. $$\int \frac{\sqrt{2au-u^2}\,du}{u^2} = -\frac{2\sqrt{2au-u^2}}{u} - \arccos\left(1-\frac{u}{a}\right) + C.$$

84. $$\int \frac{\sqrt{2au-u^2}\,du}{u^3} = -\frac{(2au-u^2)^{3/2}}{3au^3} + C.$$

85. $$\int \frac{\sqrt{2au-u^2}\,du}{u^m} = -\frac{(2au-u^2)^{3/2}}{a(2m-3)u^m} + \frac{m-3}{a(2m-3)} \int \frac{\sqrt{2au-u^2}\,du}{u^{m-1}}$$

86. $$\int \frac{du}{\sqrt{2au-u^2}} = \arccos\left(1-\frac{u}{a}\right) + C.$$

87. $$\int \frac{du}{\sqrt{2au+u^2}} = \ln(u+a+\sqrt{2au+u^2}) + C.$$

88. $$\int F(u, \sqrt{2au+u^2})\,du = \int f(z-a, \sqrt{z^2-a^2})\,dz, \text{ where } z = u+a.$$

89. $$\int \frac{u\,du}{\sqrt{2au-u^2}} = -\sqrt{2au-u^2} + a \arccos\left(1-\frac{u}{a}\right) + C.$$

90. $$\int \frac{u^2\,du}{\sqrt{2au-u^2}} = -\frac{(u+3a)\sqrt{2au-u^2}}{2} + \frac{3a^2}{2} \arccos\left(1-\frac{u}{a}\right) + C.$$

91. $$\int \frac{u^m du}{\sqrt{2au-u^2}} = -\frac{u^{m-1}\sqrt{2au-u^2}}{m} + \frac{a(2m-1)}{m} \int \frac{u^{m-1}\,du}{\sqrt{2au-u^2}}$$

92. $$\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C.$$

93. $$\int \frac{du}{u^m\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{a(2m-1)u^m} + \frac{m-1}{a(2m-1)} \int \frac{du}{u^{m-1}\sqrt{2au-u^3}}$$

94. $$\int \frac{du}{(2au-u^2)^{3/2}} = \frac{u-a}{a^2\sqrt{2au-u^2}} + C.$$

95. $$\int \frac{u\,du}{(2au-u^2)^{3/2}} = \frac{u}{a\sqrt{2au-u^2}} + C.$$

Binomial Reduction Formulas

96. $$\int u^m(a + bu^q)^p du = \frac{u^{m-q+1}(a + bu^q)^{p+1}}{b(pq + m + 1)} - \frac{a(m - q + 1)}{b(pq + m + 1)} \int u^{m-q}(a + bu^q)^p \, du.$$

97. $$\int u^m(a + bu^q)^p \, du = \frac{u^{m+1}(a + bu^q)^p}{pq + m + 1} + \frac{apq}{pq + m + 1} \int u^m (a + bu^q)^{p-1} \, du.$$

98. $$\int \frac{du}{u^m(a + bu^q)^p} = - \frac{1}{a(m - 1)u^{m-1}(a + bu^q)^{p-1}} - \frac{b(m - q + pq - 1)}{a(m - 1)} \int \frac{du}{u^{m-q}(a + bu^q)^p}$$

99. $$\int \frac{du}{u^m(a + bu^q)^p} = \frac{1}{aq(p - 1)u^{m-1}(a + bu^q)^{p-1}} + \frac{m - q + pq - 1}{aq(p - 1)} \int \frac{du}{u^m(a + bu^q)^{p-1}}$$

100. $$\int \frac{d}{u(a + bu^q)} = \frac{1}{aq} \ln \left(\frac{u^q}{a + bu^q} \right) + C.$$

101. $$\int \frac{(a + bu^q) \, du}{u^m} = - \frac{(a + bu^q)^{p+1}}{a(m - 1)u^{m-1}} - \frac{b(m - q - pq - 1)}{a(m - 1)} \int \frac{(a + bu^q)^p \, du}{u^{m-q}}$$

102. $$\int \frac{(a + bu^q)^p \, du}{u^m} = \frac{(a + bu^q)^p}{(pq - m + 1)u^{m-1}} + \frac{apq}{pq - m + 1} \int \frac{(a + bu^q)^{p-1} du}{u^m}$$

103. $$\int \frac{u^m \, du}{(a + bu^q)^p} = \frac{u^{m-q+1}}{b(m - pq + 1)(a + bu^q)^{p-1}} - \frac{a(m - q + 1)}{b(m - pq + 1)} \int \frac{u^{m-q} \, du}{(a + bu^q)^p}$$

104. $$\int \frac{u^m \, du}{(a + bu^q)^p} = \frac{u^{m+1}}{aq(p - 1)(a + bu^q)^{p-1}}$$

$$- \frac{m + q - pq + 1}{aq(p - 1)} \int \frac{u^m\, du}{(a + bu^q)^{p-1}}$$

Forms containing $a + bu \pm cu^2$ $(c > 0)$

The expression $a + bu + cu^2$ may be reduced to a binomial by writing

$$u = z - \frac{b}{2c}, \quad k = \frac{b^2 - 4ac}{4c^2}$$

Then $$a + bu + cu^2 = c(z^2 - k).$$

The expression $a + bu - cu^2$ may be reduced to a binomial by writing

$$u = z + \frac{b}{2c}, \quad k = \frac{b^2 + 4ac}{4c^2}$$

The $$a + bu - cu^2 = c(k - z^2).$$

105. $$\int \frac{du}{a + bu + cu^2} = \frac{2}{\sqrt{4ac - b^2}} \arctan\left(\frac{2cu + b}{\sqrt{4ac - b^2}}\right) + C, \text{ when } b^2 < 4ac.$$

106. $$\int \frac{du}{a + bu + cu^2} = \frac{1}{\sqrt{b^2 - 4ac}} \ln\left(\frac{2cu + b - \sqrt{b^2 - 4ac}}{2cu + b + \sqrt{b^2 - 4ac}}\right) + C, \text{ when } b^2 > 4ac.$$

107. $$\int \frac{du}{a + bu - cu^2} = \frac{1}{\sqrt{b^2 + 4ac}} \ln\left(\frac{\sqrt{b^2 + 4ac} + 2cu - b}{\sqrt{b^2 + 4ac} - 2cu + b}\right) + C.$$

108. $$\int \frac{(Mu + N)\, du}{a + bu \pm cu^2} = \frac{\pm M}{2c} \ln(a + bu \pm cu^2) + \left(N \pm \frac{bM}{2c}\right) \int \frac{du}{a + bu \pm cu^2}$$

109. $$\int \sqrt{a + bu + cu^2}\, du = \frac{2cu + b}{4c} \sqrt{a + bu + cu^2} - \frac{b^2 - 4ac}{8c^{3/2}} \ln(2cu + b + 2\sqrt{c}\sqrt{a + bu + cu^2}) + C.$$

110. $$\int \sqrt{a + bu - cu^2}\, du = \frac{2cu - b}{4c} \sqrt{a + bu - cu^2} + \frac{b^2 + 4ac}{8c^{3/2}} \arcsin\left(\frac{2cu - b}{\sqrt{b^2 + 4ac}}\right) + C.$$

111. $$\int \frac{du}{\sqrt{a + bu + cu^2}} = \frac{1}{\sqrt{c}} \ln (2cu + b + 2\sqrt{c}\sqrt{a + bu + cu^2}) + C.$$

112. $$\int \frac{du}{\sqrt{a + bu - cu^2}} = \frac{1}{\sqrt{c}} \arcsin \left(\frac{2cu - b}{\sqrt{b^2 + 4ac}} \right) + C.$$

113. $$\int \frac{u\,du}{\sqrt{a + bu + cu^2}} = \frac{\sqrt{a + bu + cu^2}}{c} - \frac{b}{2c^{3/2}} \ln (2cu + b + 2\sqrt{c}\sqrt{a + bu + cu^2}) + C.$$

114. $$\int \frac{u\,du}{\sqrt{a + bu - cu^2}} = -\frac{\sqrt{a + bu - cu^2}}{c} + \frac{b}{2c^{3/2}} \arcsin \left(\frac{2cu - b}{\sqrt{b^2 + 4ac}} \right) + C.$$

Other Algebraic Forms

115. $$\int \sqrt{\frac{a + u}{b + u}}\,du = \sqrt{(a + u)(b + u)} + (a - b) \log. (\sqrt{a + u} + \sqrt{b + u}) + C.$$

116. $$\int \sqrt{\frac{a - u}{b - u}}\,du = \sqrt{(a - u)(b + u)} + (a + b) \arcsin \sqrt{\frac{u + b}{a + b}} + C.$$

117. $$\int \sqrt{\frac{a + u}{b - u}}\,du = \sqrt{(a + u)(b - u)} - (a + b) \arcsin \sqrt{\frac{b - u}{a + b}} + C.$$

118. $$\int \sqrt{\frac{1 + u}{1 - u}}\,du = \sqrt{1 - u^2} + \arcsin u + C.$$

119. $$\int \frac{du}{\sqrt{(u - a)(b - u)}} = 2 \arcsin \sqrt{\frac{u - a}{b - a}} + C.$$

Exponential and Logarithmic Forms

120. $\int e^{au}\,du = \frac{e^{au}}{a} + C.$

121. $\int b^{au}\,du = \frac{b^{au}}{a \ln b} + C.$

122. $\int ue^{au}\,du = \frac{e^{au}}{a^2}(au - 1) + C.$

123. $\int u^n e^{au}\,du = \frac{u^n e^{au}}{a} - \frac{n}{a}\int u^{n-1} e^{au}\,du$

124. $\int u^n b^{au}\,du = \frac{u^n b^{au}}{a \ln b} - \frac{n}{a \ln b}\int u^{n-1} b^{au}\,du + C.$

125. $\int \frac{b^{au}du}{u^n} = -\frac{b^{au}}{(n-1)u^{n-1}} + \frac{a \ln b}{n-1}\int \frac{b^{au}\,du}{u^{n-1}}$

126. $\int \ln u\,du = u \ln u - u + C.$

127. $\int u^n \ln u\,du = u^{n+1}\left[\frac{\ln u}{n+1} - \frac{1}{(n+1)^2}\right] + C.$

128. $\int u^m \ln^n u\,du = \frac{u^{m+1}}{m+1}\ln^n u - \frac{n}{m+1}\int u^m \ln^{n-1} u\,du.$

129. $\int e^{au} \ln u\,du = \frac{e^{au} \ln u}{a} - \frac{1}{a}\int \frac{e^{au}}{u}\,du.$

130. $\int \frac{du}{u \ln u} = \ln(\ln u) + C.$

Trigonometric Forms

In forms involving tan u, ctn u, sec u, csc u, which do not appear below, first use the relations

$$\tan u = \frac{\sin u}{\cos u},\ \operatorname{ctn} u = \frac{\cos u}{\sin u},\ \sec u = \frac{1}{\cos u},\ \csc u = \frac{1}{\sin u}$$

131. $\int \sin u\,du = -\cos u + C.$

132. $\int \cos u \, du = \sin u + C.$

133. $\int \tan u \, du = -\ln \cos u + C = \ln \sec u + C.$

134. $\int \operatorname{ctn} u \, du = \ln \sin u + C.$

135. $\int \sec u \, du = \int \frac{du}{\cos u} = \ln (\sec u + \tan u) + C.$

$$= \ln \tan \left(\frac{u}{2} + \frac{\pi}{4} \right) + C.$$

136. $\int \csc u \, du = \int \frac{du}{\sin u} = \ln (\csc u - \operatorname{ctn} u) + C$

$$= \ln \tan \frac{u}{2} + C.$$

137. $\int \sec^2 u \, du = \tan u + C.$

138. $\int \csc^2 u \, du = -\operatorname{ctn} u + C.$

139. $\int \sec u \tan u \, du = \sec u + C.$

140. $\int \csc u \operatorname{ctn} u \, du = -\csc u + C.$

141. $\int \sin^2 u \, du = \frac{1}{2} u - \frac{1}{4} \sin 2u + C.$

142. $\int \cos^2 u \, du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C.$

143. $\int \cos^n u \sin u \, du = -\frac{\cos^{n+1} u}{n + 1} + C.$

144. $\int \sin^n u \cos u \, du = \frac{\sin^{n+1} u}{n + 1} + C.$

145. $\int \sin mu \sin nu \, du = -\frac{\sin (m + n) u}{2(m + n)} + \frac{\sin (m - n)u}{2(m - n)} + C.$

146. $\int \cos mu \cos nu \, du = \frac{\sin (m + n)u}{2(n + n)} + \frac{\sin (m - n)u}{2(m - n)} + C.$

147. $\int \sin mu \cos nu \, du = -\frac{\cos (m + n)u}{2(m + n)} - \frac{(\cos m - n)u}{2(m - n)} + C.$

148. $\int \frac{du}{1 + \cos a \cos u} = 2 \csc a \arctan (\tan \frac{1}{2} a \tan \frac{1}{2} u) + C.$

149. $$\int \frac{du}{\cos a + \cos u} = \csc a \ln \left(\frac{1 + \tan \frac{1}{2} a \tan \frac{1}{2} u}{1 - \tan \frac{1}{2} a \tan \frac{1}{2} u} \right) + C$$

$$(\tan^2 \tfrac{1}{2} u < \operatorname{ctn}^2 \tfrac{1}{2} a)$$

$$= 2 \csc a \tanh^{-1} (\tan \tfrac{1}{2} a \tan \tfrac{1}{2} u) + C \quad (\tan^2 \tfrac{1}{2} u < \operatorname{ctn}^2 \tfrac{1}{2} a)$$

150. $$\int \frac{du}{1 + \cos a \sin u} = 2 \csc a \arctan (\csc a \tan \tfrac{1}{2} u + \operatorname{ctn} a) + C.$$

151. $$\int \frac{du}{\cos a + \sin u} = \csc a \operatorname{lin} \left(\frac{\tan a - \tan \frac{1}{2} u - \sec a}{\tan a + \tan \frac{1}{2} u + \sec a} \right) + C.$$

$$[(\operatorname{ctn} a \tan \tfrac{1}{2} u + \csc a)^2 < 1]$$

$$= -2 \csc a \tanh^{-1} (\operatorname{ctn} a \tan \tfrac{1}{2} u + \csc a) + C$$

$$[(\operatorname{ctn} a \tan \tfrac{1}{2} u + \csc a)^2 < 1]$$

152. $$\int \frac{du}{a^2 \cos^2 u + b^2 \sin^2 u} = \frac{1}{ab} \arctan \left(\frac{b \tan u}{a} \right) + C.$$

153. $$\int e^{au} \sin nu \, du = \frac{e^{au} (a \sin nu - n \cos nu)}{a^2 + n^2} + C.$$

154. $$\int e^{au} \cos nu \, du = \frac{e^{au} (n \sin nu + a \cos nu)}{a^2 + n^2} + C.$$

155. $$\int u \sin u \, du = \sin u - u \cos u + C.$$

156. $$\int u \cos u \, du = \cos u + u \sin u + C.$$

Trigonotric Reduction Formulas

157. $$\int \sin^n u \, du = -\frac{\sin^{n-1} u \cos u}{n} + \frac{n-1}{n} \int \sin^{n-2} u \, du.$$

158. $$\int \cos^n u \, du = \frac{\cos^{n-1} u \sin u}{n} + \frac{n-1}{n} \int \cos^{n-2} u \, du.$$

159. $\int \frac{du}{\sin^n u} = -\frac{\cos u}{(n-1)\sin^{n-1} u} + \frac{n-2}{n-1}\int \frac{du}{\sin^{n-2} u}$

160. $\int \frac{du}{\cos^n u} = \frac{\sin u}{(n-1)\cos^{n-1} u} + \frac{n-2}{n-1}\int \frac{du}{\cos^{n-2} u}$

161. $\int \cos^m u \sin^n u\, du = \frac{\cos^{m-1} u \sin^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2} u \sin^n u\, du.$

162. $\int \cos^m u \sin^n u\, du = -\frac{\sin^{n-1} u \cos^{m+1} u}{m+n} + \frac{n-1}{m+n}\int \cos^m u \sin^{n-2} u\, du.$

163. $\int \frac{du}{\cos^m u \sin^n u} = \frac{1}{(m-1)\sin^{n-1} u \cos^{m-1} u} + \frac{m+n-2}{m-1}\int \frac{du}{\cos^{m-2} u \sin^n u}$

164. $\int \frac{du}{\cos^m u \sin^n u} = -\frac{1}{(n-1)\sin^{n-1} u \cos^{m-1} u} + \frac{m+n-2}{n-1}\int \frac{du}{\cos^m u \sin^{n-2} u}$

165. $\int \frac{\cos^m u\, du}{\sin^n u} = -\frac{\cos^{m+1} u}{(n-1)\sin^{n-1} u} - \frac{m-n+2}{n-1}\int \frac{\cos^m u\, du}{\sin^{n-2} u}$

166. $\int \frac{\cos^m u\, du}{\sin^n u} = \frac{\cos^{m-1} u}{(m-n)\sin^{n-1} u} + \frac{m-1}{m-n}\int \frac{\cos^{m-2} u\, du}{\sin^n u}$

167. $\int \frac{\sin^n u\, du}{\cos^m u} = \frac{\sin^{n+1} u}{(m-1)\cos^{m-1} u} - \frac{n-m+2}{m-1}\int \frac{\sin^n u\, du}{\cos^{m-2} u}$

168. $\int \frac{\sin^n u\, du}{\cos^m u} = -\frac{\sin^{n-1} u}{(n-m)\cos^{m-1} u} + \frac{n-1}{n-m}\int \frac{\sin^{n-2} u\, du}{\cos^m u}$

169. $\int \tan^n u\, du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\, du.$

170. $\int \operatorname{ctn}^n u\, du = \frac{\operatorname{ctn}^{n-1} u}{n-1} - \int \operatorname{ctn}^{n-2} u\, du.$

171. $\int e^{au} \cos^n u\, du = \frac{e^{au} \cos^{n-1} u(a \cos u + n \sin u)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2}\int e^{au} \cos^{n-2} u\, du.$

172. $\int e^{au} \sin^n u \, du = \frac{e^{au} \sin^{n-1} u(a \sin u - n \cos u)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} \int e^{au} \sin^{n-2} u \, du.$

173. $\int u^m \cos au \, du = \frac{u^{m-1}}{a^2} (au \sin au + m \cos au) - \frac{m(m-1)}{a^2} \int u^{m-2} \cos au \, du.$

174. $\int u^m \sin au \, du = \frac{u^{m-1}}{a^2} (m \sin au - au \cos au) - \frac{m(m-1)}{a^2} \int u^{m-2} \sin au \, du.$

Inverse Trigonometric Functions

175. $\int \text{arc sin } u \, du = u \text{ arc sin } u + \sqrt{1 - u^2} + C.$

176. $\int \text{arc cos } u \, du = u \text{ arc cos } u - \sqrt{1 - u^2} + C.$

177. $\int \text{arc tan } u \, du = u \text{ arc tan } u - \ln \sqrt{1 + u^2} + C.$

178. $\int \text{arc ctn } u \, du = u \text{ arc ctn } u + \ln \sqrt{1 + u^2} + C.$

179. $\int \text{arc sec } u \, du = u \text{ arc sec } u - \ln (u + \sqrt{u^2 - 1}) + C.$

$= u \text{ arc sec } u - \cosh^{-1} u + C.$

180. $\int \text{arc csc } u \, du = u \text{ arc csc } u + \ln (u + \sqrt{u^2 - 1}) + C$

$= u \text{ arc csc } u + \cosh^{-1} u + C.$

Hyperbolic Functions

181. $\int \sinh u \, du = \cosh u + C.$

182. $\int \cosh u \, du = \sinh u + C.$

183. $\int \tanh u \, du = \ln \cosh u + C.$

184. $\int \operatorname{ctnh} u \, du = \ln \sinh u + C.$

185. $\int \operatorname{sech} u \, du = \arctan(\sinh u) + C$

186. $\int \operatorname{csch} u \, du = \ln \tanh \frac{1}{2} u + C.$

187. $\int \operatorname{sech}^2 u \, du = \tanh u + C.$

188. $\int \operatorname{csch}^2 u \, du = -\operatorname{ctnh} u + C.$

189. $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C.$

190. $\int \operatorname{csch} u \operatorname{ctnh} u \, du = -\operatorname{csch} u + C.$

191. $\int \sinh^2 u \, du = \frac{1}{4} \sinh 2u - \frac{1}{2} u + C.$

192. $\int \cosh^2 u \, du = \frac{1}{4} \sinh 2u + \frac{1}{2} u + C.$

193. $\int \tanh^2 u \, du = u - \tanh u + C.$

194. $\int \operatorname{ctnh}^2 u \, du = u - \operatorname{ctnh} u + C.$

195. $\int u \sinh u \, du = u \cosh u - \sinh u + C.$

196. $\int u \cosh u \, du = u \sinh u - \cosh u + C.$

197. $\int \sinh^{-1} u \, du = u \sinh^{-1} u - \sqrt{1 + u^2} + C.$

198. $\int \cosh^{-1} u \, du = u \cosh^{-1} u - \sqrt{u^2 - 1} + C.$

199. $\int \tanh^{-1} u \, du = u \tanh^{-1} u + \frac{1}{2} \ln (1 - u^2) + C.$

200. $\int \operatorname{ctnh}^{-1} u \, du = u \operatorname{ctnh}^{-1} u + \frac{1}{2} \ln (1 - u^2) + C.$

201. $\int \operatorname{sech}^{-1} u \, du = u \operatorname{sech}^{-1} u + \operatorname{gd}(\tanh^{-1} u) + C$

$= u \operatorname{sech}^{-1} u + \arcsin u + C.$

202. $\int \operatorname{csch}^{-1} u \, du = u \operatorname{csch}^{-1} u + \sinh^{-1} u + C.$

203. $\int \sinh mu \sinh nu\, du = \frac{\sinh (m + n)u}{2(m + n)} - \frac{\sinh (m - n)u}{2(m - n)} + C. \left(m \gtrless n\right)$

204. $\int \cosh mu \cosh nu\, du = \frac{\sinh (m + n)\, u}{2\,(m + n)} + \frac{\sinh (m - n)\, u}{2\,(m - n)} + C. \left(m \gtrless n\right)$

205. $\int \sinh mu \cosh nu\, du = \frac{\cosh (m + n)\, u}{2\,(m + n)} + \frac{\cosh (m - n)\, u}{2\,(m - n)} + C. \left(m \gtrless n\right)$

206. $\int \frac{du}{\cos a + \cosh u} = 2 \operatorname{csch} a \tanh^{-1} \left(\tanh \frac{1}{2} u \tan \frac{1}{2} a\right) + C.$

207. $\int \frac{du}{\cos a + \cosh u} = 2 \csc a \arctan \left(\tanh \frac{1}{2} u \tan \frac{1}{2} a\right) + C.$

208. $\int \frac{du}{1 + \cos a \cosh u} = 2 \csc a \tanh^{-1} \left(\tanh \frac{1}{2} u \tan \frac{1}{2} a\right) + C.$

$\left(\tanh^2 \frac{1}{2} u < \operatorname{ctn}^2 \frac{1}{2} a\right)$

209. $\int e^{au} \sinh nu\, du = \frac{e^{au}(a \sinh nu - n \cosh nu)}{a^2 - n^2} + C.$

210. $\int e^{au} \cosh nu\, du = \frac{e^{au}\,(a \cosh nu - n \sinh nu)}{a^2 - n^2} + C.$