\[
\int_0^1 x \sin n\pi x \, dx = x \left( -\frac{\cos n\pi x}{n\pi} \right) \bigg|_0^1 - \int_0^1 \left( -\frac{\cos n\pi x}{n\pi} \right) \, dx
\]

\[
= -\frac{\cos n\pi}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2} \bigg|_0^1
\]

\[
= -\frac{(-1)^n}{n\pi} + \frac{1}{n^2\pi^2} (\sin n\pi - 0)
\]

\[
= \frac{-(-1)^n}{n\pi}
\]

(2)

และ \[
\int_0^1 x^3 \sin n\pi x \, dx = x^3 \left( -\frac{\cos n\pi x}{n\pi} \right) \bigg|_0^1 - \int_0^1 \left( -\frac{\cos n\pi x}{n\pi} \right) 3x^2 \, dx
\]

\[
= -\frac{\cos n\pi}{n\pi} + \frac{3}{n\pi} \int_0^1 x^2 \cos n\pi x \, dx
\]

อินทิกรัจสีและส่วนอีกครั้ง

\[
= -\frac{(-1)^n}{n\pi} + \frac{3}{n\pi} \left[ x \frac{\sin n\pi x}{n\pi} \right]_0^1
\]

\[
= -\frac{(-1)^n}{n\pi} + \frac{3}{n\pi} \left[ 0 - \frac{2}{n\pi} \int_0^1 \sin n\pi x \, dx \right]
\]

\[
= \frac{(-1)^n}{n\pi} - \frac{6}{n^2\pi^2} \int_0^1 \sin n\pi x \, dx
\]

แต่ \[
\int_0^1 x \sin nx \, dx = \frac{-(-1)^n}{n\pi}
\]

เพราะว่า \(1\) ค่า \(= 2\pi \times \frac{1}{2} = 4\) เพราะฉะนั้น \(l = 2\)

หาค่า \(a_0\) จากสูตร

\[
a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \, dx
\]

\[
= \frac{1}{2} \int_{-2}^{2} f(x) \, dx
\]

\[
= \frac{1}{2} \left[ \int_{-2}^{-1} (0) \, dx + \int_{-1}^{0} (1) \, dx + \int_{0}^{1} (-1) \, dx + \int_{1}^{2} (0) \, dx \right]
\]
\[ = \frac{1}{2} \left[ 0 + \int_{-1}^{0} x \, dx \right] \]
\[ = \frac{1}{2} \left[ 1 - 1 \right] \]
\[ = 0 \]

หาค่า \( a_n \) จากสูตร

\[ a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx \]
\[ = \frac{1}{2} \int_{-1}^{1} f(x) \cos \frac{n\pi x}{2} \, dx \]
\[ = \frac{1}{2} \left[ \int_{-1}^{0} f(x) \cos \frac{n\pi x}{2} \, dx + \int_{0}^{1} f(x) \cos \frac{n\pi x}{2} \, dx \right] \]
\[ = \frac{1}{2} \left[ \int_{-1}^{0} \cos \frac{\pi}{1} \, dx + \int_{0}^{1} \cos \frac{\pi}{0} \, dx \right] \]
\[ = \frac{1}{2} \left[ 0 + \int_{-1}^{0} \cos \frac{n\pi x}{2} \, dx \right] \]
\[ = \frac{1}{2} \left[ 0 + \frac{\sin \frac{n\pi}{2}}{n\pi} \right] \]
\[ = 0 \]

หาค่า \( b_n \) จากสูตร

\[ b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx \]
\[ = \frac{1}{2} \int_{-1}^{1} f(x) \sin \frac{n\pi x}{2} \, dx \]

คิดถึง

\[ \int_{0}^{1} x^3 \sin n\pi x \, dx = \frac{(-1)^n}{n\pi} \frac{6}{n^2} \pi \left[ \frac{(-1)^n}{n\pi} \right] \]
แทนค่า (2) และ (3) ใน (1) จะได้

\[
b_n = 2 \left[ \frac{\frac{-4}{\pi} \frac{-1}{n}}{1} - \frac{\frac{6}{\pi} \frac{-1}{n}}{1} + \frac{\frac{6}{\pi} \frac{-1}{n}}{1} \right]
\]

\[
= \frac{-2}{\pi} \frac{-1}{n} + \frac{2}{\pi} \frac{-1}{n} \frac{12(-1)^n}{n^3\pi^3}
\]

\[
= \frac{-12(-1)^n}{n^3\pi^3}
\]

ตั้งนั้น แทนค่า \(a_0, a_n\) และ \(b_n\) ในสูตรอนุกรมฟูเรียร์

\[
f(x) = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3\pi^3} \sin n\pi x
\]

หรือ

\[
x - x^3 = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin n\pi x
\]

11. \(f(x)\) =

\[
\begin{cases}
0 ; & -2 < x < -1 \\
1 ; & -1 < x < 0 \\
-1 ; & 0 < x < 1 \\
1 ; & 1 < x < 2
\end{cases}
\]

วิธีทำ เขียนกราฟของฟังก์ชัน \(f(x)\)

สูตรอนุกรมฟูเรียร์ คือ

\[
f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)
\]

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\[
\begin{align*}
= & \frac{1}{2} \left[ \int_{-2}^{-1} (0) \sin \frac{n\pi x}{2} \, dx + \int_{-1}^{0} (1) \sin \frac{n\pi x}{2} \, dx + \int_{0}^{1} (-1) \sin \frac{n\pi x}{2} \, dx \right] \\
& + \left[ \int_{1}^{2} (0) \sin \frac{n\pi x}{2} \, dx \right]
\end{align*}
\]

\[
= \frac{1}{2} \left[ 0 - \left. \cos \frac{n\pi x}{2} \right|_{-1}^{0} + \left. \cos \frac{n\pi x}{2} \right|_{0}^{1} + 0 \right]
\]

\[
= \frac{1}{2} \left[ -2 \cos \left( \frac{-n\pi}{2} \right) \right] + \frac{2}{n\pi} \left[ \cos \frac{n\pi}{2} - 1 \right]
\]

\[
= \frac{1}{2} \left( \frac{2}{n\pi} \right) \left[ -1 + \cos \frac{n\pi}{2} + \cos \frac{n\pi}{2} - 1 \right]
\]

\[
= \frac{1}{n\pi} \left[ 2 \cos \frac{n\pi}{2} - 2 \right]
\]

\[
= \frac{2}{n\pi} \left[ \cos \frac{n\pi}{2} - 1 \right]
\]

แทนค่า \(a_0, a_n\) และ \(b_n\) ดังในสูตรอนุมการฟูเรียร์

\[
f(x) = \frac{1}{2} (0) + \sum_{n=1}^{\infty} \left[ (0) \cos \frac{n\pi x}{2} + \frac{2}{n\pi} \left( \cos \frac{n\pi}{2} - 1 \right) \sin \frac{n\pi x}{2} \right]
\]

\[
= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \cos \frac{n\pi}{2} - 1 \right) \sin \frac{n\pi x}{2}
\]

12. \(f(x) = \begin{cases} 
0 & ; \quad -2 < x < -1 \\
1 + x & ; \quad -1 < x < 0 \\
1 - x & ; \quad 0 < x < 1 \\
0 & ; \quad 1 < x < 2
\end{cases}\)

วิธีทำ เขียนกราฟ

\[
\begin{array}{c}
\text{f(x)} \\
\hline
-2 & -1 & 0 & 1 & 2 \\
\hline
\end{array}
\]

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MA 446 (H)
เพราะ 1 สาม = 2τ = 4 เหรียญ 2 เหรียญ

จากกราฟพบว่า ฟ(x) เป็นฟังก์ชันต่อ นั่นคือ บₙ = 0

ดังนั้น สรุปการหาปริยว คือ

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} \]

\[ a_0 = \frac{2}{\ell} \int_{0}^{\ell} f(x) \, dx \]

\[ = \frac{2}{\ell} \int_{0}^{\ell} f(x) \, dx \]

\[ = \int_{0}^{1} (1 - x) \, dx + \int_{1}^{2} (0) \, dx \]

\[ = \left( x - \frac{x^2}{2} \right) \bigg|_{0}^{1} + 0 \]

\[ = 1 - \frac{1}{2} \]

\[ = \frac{1}{2} \]

จากสูตร

\[ a_n = \frac{2}{\ell} \int_{0}^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx \]

\[ = \frac{2}{\ell} \int_{0}^{\ell} f(x) \cos \frac{n\pi x}{2} \, dx \]

\[ = \int_{0}^{1} (1 - x) \cos \frac{n\pi x}{2} \, dx + \int_{1}^{2} (0) \cos \frac{n\pi x}{2} \, dx \]

\[ = \int_{0}^{1} \cos \frac{n\pi x}{2} \, dx - \int_{0}^{1} x \cos \frac{n\pi x}{2} \, dx + 0 \]

\[ = \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \bigg|_{0}^{1} - \left\{ x \left( \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) \bigg|_{0}^{1} - \int_{0}^{1} \left( \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) \, dx \right\} \]

\[ = \frac{2}{n\pi} \sin \frac{n\pi}{2} - \left\{ \frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} \bigg|_{0}^{1} \right\} \]
\[ = \frac{2}{n\pi} \sin \frac{n\pi}{2} - \frac{2}{n\pi} \sin \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \left[ \cos \frac{n\pi}{2} - 1 \right] \]

\[ = \frac{4}{n^2\pi^2} \left[ 1 - \cos \frac{n\pi}{2} \right] \]

แทนค่าในสูตรอนุกรมฟูเรียร์

\[ f(x) = \frac{1}{2} \left( \frac{1}{2} \right) + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( 1 - \cos \frac{n\pi}{2} \right) \cos \frac{n\pi x}{2} \]

\[ = \frac{1}{4} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( 1 - \cos \frac{n\pi}{2} \right) \cos \frac{n\pi x}{2} \]
เล่ยแบบฝึกหัด 2.4

1. จงแสดงว่า ถ้า

\[ f(x) = \begin{cases} 
  x & \text{สำหรับ } 0 < x < \frac{\pi}{2} \\
  \pi - x & \text{สำหรับ } \frac{\pi}{2} < x < \pi 
\end{cases} \]

tั้งนี้

\[ f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left( \frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots \right) \]

วิธีทำ เขียนกราฟของ \( f(x) \)

\[
\begin{aligned}
 f(x) &= 1 - a_0 + \frac{1}{\pi} \sum_{n=1}^{\infty} \left( a_n \cos \frac{nx}{\ell} + b_n \sin \frac{nx}{\ell} \right) \\
  a_0 &= \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \, dx \\
  a_0 &= \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} f(x) \, dx
\end{aligned}
\]

เพราะว่า \( \ell = 2\ell = \pi \) เพราะฉะนั้น \( \ell = \frac{\pi}{2} \) เลือกค่า \( c = 0 \) ตั้งนั้น

\[
\begin{aligned}
  a_0 &= \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} f(x) \, dx \\
  &= \frac{2}{\pi} \left( \int_{0}^{\frac{\pi}{2}} f(x) \, dx \right)
\end{aligned}
\]

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\[ a_n = \frac{1}{\ell} \int_0^{c+2\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx \]

\[ a_n = \frac{1}{\pi} \int_0^{\pi/2} f(x) \cos 2nx \, dx \]

\[ a_n = \frac{2}{\pi} \int_0^{\pi/2} f(x) \cos 2nx \, dx \]

\[ a_n = \frac{2}{\pi} \left[ \int_0^{\pi/2} x \cos 2nx \, dx + \pi \int_{\pi/2}^{\pi} (\pi - x) \cos 2nx \, dx \right] \]

\[ a_n = \frac{2}{\pi} \left[ \int_0^{\pi/2} x \cos 2nx \, dx + \pi \int_{\pi/2}^{\pi} \cos 2nx \, dx - \int_{\pi/2}^{\pi} x \cos 2nx \, dx \right] \]

\[ \int_0^{\pi/2} x \cos 2nx \, dx = \int_0^{\pi/2} \left( \frac{\sin 2nx}{2n} \right) \, dx - \int_0^{\pi/2} \left( \frac{\sin 2nx}{2n} \right) \, dx \]

\[ = 0 + \cos 2nx \cdot \frac{\pi}{4n^2} \]

\[ \frac{\pi}{2} \]

ทว่า \( a_n \) จากสูตร

\[ a_n = \frac{1}{\ell} \int_0^{c+2\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx \]

\[ a_n = \frac{1}{\pi} \int_0^{\pi/2} f(x) \cos 2nx \, dx \]

\[ a_n = \frac{2}{\pi} \int_0^{\pi/2} f(x) \cos 2nx \, dx \]

\[ a_n = \frac{2}{\pi} \left[ \int_0^{\pi/2} x \cos 2nx \, dx + \pi \int_{\pi/2}^{\pi} (\pi - x) \cos 2nx \, dx \right] \]

\[ a_n = \frac{2}{\pi} \left[ \int_0^{\pi/2} x \cos 2nx \, dx + \pi \int_{\pi/2}^{\pi} \cos 2nx \, dx - \int_{\pi/2}^{\pi} x \cos 2nx \, dx \right] \]

\[ \int_0^{\pi/2} x \cos 2nx \, dx = \int_0^{\pi/2} \left( \frac{\sin 2nx}{2n} \right) \, dx - \int_0^{\pi/2} \left( \frac{\sin 2nx}{2n} \right) \, dx \]

\[ = 0 + \cos 2nx \cdot \frac{\pi}{4n^2} \]

\[ \frac{\pi}{2} \]
\[
\int_{\pi/2}^{\pi} \cos 2nx \, dx = \frac{\sin 2nx}{2n} \bigg|_{\pi/2}^{\pi} = \frac{1}{2n} (\sin 2n\pi - \sin n\pi)
\]
\[
= \frac{1}{2n} (0) ; \quad \sin 2n\pi = \sin n\pi = 0
\]
\[
= 0 \quad \ldots \ldots (3)
\]

And
\[
\int_{\pi/2}^{\pi} x \cos 2nx \, dx = x \left( \frac{\sin 2nx}{2n} \right) \bigg|_{\pi/2}^{\pi} - \frac{\sin 2nx}{2n} \bigg|_{\pi/2}^{\pi} \]
\[
= \frac{1}{2n} \left[ \pi \sin 2n\pi - \frac{\pi}{2} \sin n\pi \right] + \cos 2nx \bigg|_{\pi/2}^{\pi} \]
\[
= \frac{1}{2n} (0) + \frac{1}{4n^2} \left[ \cos 2n\pi - \cos n\pi \right]
\]
\[
= -\frac{1}{4n^2} - \frac{(-1)^n}{4n^2} \quad \ldots \ldots (4)
\]

Then from (2), (3) and (4) putting (1) will give

\[
a_n = \frac{2}{\pi} \left[ \frac{(-1)^n}{4n^2} - \frac{1}{4n^2} + \pi (0) - \frac{1}{4n^2} + \frac{(-1)^n}{4n^2} \right]
\]
\[
= \frac{2}{\pi} \left[ \frac{(-1)^n}{2n^2} - \frac{1}{2n^2} \right]
\]
\[
= \frac{(-1)^n}{\pi n^2}
\]

For \(b_n\) from the formula

\[
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx
\]

Then for \(\ell = \frac{\pi}{2}\) and \(c = 0\) will give

\[\text{MA 446 (H)}\]
\[ b_n = \frac{1}{\pi} \int_0^{\pi} f(x) \sin 2nx \, dx \]
\[ = \frac{2}{\pi} \int_0^{\pi} f(x) \sin 2nx \, dx \]
\[ = \frac{2}{\pi} \left[ \int_0^{\pi/2} (x) \sin 2nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin 2nx \, dx \right] \]
\[ = \frac{2}{\pi} \left[ \int_0^{\pi/2} x \sin 2nx \, dx + \int_{\pi/2}^{\pi} \sin 2nx \, dx - \int_{\pi/2}^{\pi} x \sin 2nx \, dx \right] \]

\[ \text{พิจารณา} \]
\[ \int_{0}^{\pi/2} x \sin 2nx \, dx = x \left( -\frac{\cos 2nx}{2n} \right) \bigg|_0^{\pi/2} - \int_0^{\pi/2} \left( -\frac{\cos 2nx}{2n} \right) \, dx \]
\[ = \frac{-1}{2n} \left\{ \frac{\pi}{2} \cos nx \right\} + \frac{\sin 2nx}{4n^2} \bigg|_0^{\pi/2} \]
\[ = \frac{-\pi}{4n} \left( -1 \right)^n + \frac{1}{4n^2} \left\{ \sin n\pi - 0 \right\} \]
\[ = \frac{-\pi}{4n} \left( -1 \right)^n \]
\[ \text{...........(6)} \]
\[ \int_{\pi/2}^{\pi} \sin 2nx \, dx = -\frac{\cos 2nx}{2n} \bigg|_{\pi/2}^{\pi} \]
\[ = \frac{-1}{2n} \left( \cos 2n\pi - \cos n\pi \right) \]
\[ = \frac{-1}{2n} + \frac{(-1)^n}{2n} ; \quad \cos 2n\pi = 1 \]
\[ \text{...........(7)} \]

\[ \text{และ} \quad \int_{\pi/2}^{\pi} x \sin 2nx \, dx = x \left( -\frac{\cos 2nx}{2n} \right) \bigg|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \left( -\frac{\cos 2nx}{2n} \right) \, dx \]
\[ = \frac{-1}{2n} \left\{ \pi \cos 2n\pi - \frac{\pi}{2} \cos n\pi \right\} + \frac{\sin 2nx}{4n^2} \bigg|_{\pi/2}^{\pi} \]
\[ = \frac{-\pi}{2n} + \frac{\pi}{4n} \left( -1 \right)^n + \frac{1}{4n^2} \left\{ \sin 2n\pi - \sin n\pi \right\} \]
\[
\frac{-\pi}{2n} + \frac{\pi(-1)^n}{4n} = \ldots \ldots \ldots (8)
\]

แทนค่า (6), (7) และ (6) ลงใน (5) จะได้

\[
b_n = \frac{2}{\pi} \left[ \frac{-\pi(-1)^n}{4n} + \pi \left\{ \frac{-1}{2n} + \frac{(-1)^n}{2n} \right\} - \left\{ \frac{-\pi}{2n} + \frac{\pi(-1)^n}{4n} \right\} \right]
\]

\[
b_n = 0
\]

แทนค่า \(a_0, a_n\) และ \(b_n\) ลงในสูตรอนุกรมฟูเรียร์

\[
f(x) = \frac{1}{2} \left( \frac{\pi}{2} \right) + \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n - 1}{\pi n^2} \cos 2nx + (0) \sin 2nx \right\}
\]

\[
f(x) = \frac{\pi}{4} + \frac{1}{\pi} \left\{ \frac{-2}{1^2} \cos 2x + 0 + \frac{(-2)}{3^2} \cos 6x + 0 + \ldots \right\}
\]

\[
f(x) = \frac{n}{4} - \frac{2}{\pi} \left( \frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots \right)
\]

จากข้อ 2 เพื่อข้อ 9 จะหาอนุกรมฟูเรียร์ของฟังก์ชัน \(f(x)\) ซึ่งมีค่าเป็น \(T\) ในมีอักษรค่า

นิยามเป็น

2. \(f(x) = x\); \(0 < x < 3\), \(T = 3\)

วิธีทำ สูตรอนุกรมฟูเรียร์ คือ

\[
f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)
\]

เพราะว่า 1 ค่า = \(2\ell = 3\) เพราะว่า \(\ell = \frac{3}{2}\) หาค่า \(a_0\)

จากสูตร

\[
a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \, dx
\]

เลือก \(c = 0\) ดังนี้

\[
a_0 = \frac{1}{3} \int_{0}^{\frac{3}{2}} (x) \, dx
\]

\[
= \frac{1}{3} \left[ \frac{x^2}{2} \right]_{0}^{\frac{3}{2}} = \frac{1}{3} \left( \frac{3}{2} \right)^2 = \frac{3}{4}
\]

\[
MA 446 (H)
\]

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\[ = \frac{2}{3} \int_{0}^{3} x \, dx \]
\[ = \frac{2}{3} \left( \frac{x^2}{2} \right) \bigg|_{0}^{3} \]
\[ = 3 \]

หาค่า \( a_n \) จากสูตร

\[
a_n = \frac{1}{\ell} \int_{c}^{c+2\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx
\]
\[ = \frac{1}{3} \int_{0}^{3} f(x) \cos \frac{3}{2} \, dx
\]
\[ = \frac{2}{3} \int_{0}^{3} x \cos \frac{2n\pi x}{3} \, dx
\]

อินทิเกตฟังก์ชัน

\[
= \frac{2}{3} \left[ x \left( \frac{\sin \frac{2n\pi x}{3}}{\frac{2n\pi}{3}} \right) \bigg|_{0}^{3} - \int_{0}^{3} \left( \frac{\sin \frac{2n\pi x}{3}}{\frac{2n\pi}{3}} \right) \, dx \right]
\]
\[ = \frac{2}{3} \left[ \frac{3}{2n\pi} \left\{ 3 \sin 2n\pi - 0 \right\} + \frac{9}{4n^2\pi^2} \cos \frac{2n\pi x}{3} \bigg|_{0}^{3} \right]
\[ = \frac{2}{3} \left[ 0 + \frac{9}{4n^2\pi^2} \left( \cos 2n\pi - 1 \right) \right]
\[ = \frac{2}{3} \left[ \frac{9}{4n^2\pi^2} \left( 1 - 1 \right) \right]
\[ = 0
\]

หาค่า \( b_n \) จากสูตร

\[
b_n = \frac{1}{\ell} \int_{c}^{c+2\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx
\]
\[ = \frac{2}{3} \int_{0}^{3} x \sin \frac{2n\pi x}{3} \, dx
\]
\[ = \frac{2}{3} \left[ x \left( -\cos \frac{2n\pi x}{3} \right) \bigg|_{0}^{3} - \int_{0}^{3} \left( -\cos \frac{2n\pi x}{3} \right) \, dx \right] \]

\[ = \frac{2}{3} \left[ -3 \cos 2n\pi - 1 \right] + \frac{9}{4n^2\pi^2} \sin \frac{2n\pi x}{3} \bigg|_{0}^{3} \]

\[ = -\frac{2}{n\pi} \]

แทนค่าในสูตรอนุกรมฟูเรียร์

\[ f(x) = \frac{1}{2} (3) + \frac{\cos 2n\pi x}{3} + \left( \frac{2n\pi x}{3} \right) \sin \frac{2n\pi x}{3} \bigg] \]

\[ = \frac{3}{2} - \frac{2n\pi}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{3} \]

3. \( f(x) = x^2; \ 0 < x < 2; \ T = 2 \)

วิธีทำ พิจารณาค่า \( 1 \) ทำบุญ \( 2 \ell = 2 \) เพราะฉะนั้น \( \ell = 1 \) สูตรอนุกรมฟูเรียร์ คือ

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \]

หาค่า \( a_0 \) จากสูตร

\[ a_0 = \frac{1}{\ell} \int_{0}^{2\ell} f(x) \, dx \]

เลือกค่า \( c = 0 \) ดังนั้น

\[ a_0 = \frac{1}{\ell} \int_{0}^{2\ell} x^2 \, dx \]

\[ = \frac{8}{3} \]

หาค่า \( a_n \) จากสูตร

\[ a_n = \frac{1}{\ell} \int_{c}^{c+2\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx \]
\begin{align*}
&= \frac{1}{1} \int_{0}^{2 \pi} (x^2) \cos \frac{n \pi x}{\pi} \, dx \\
&= \int_{0}^{2 \pi} x^2 \cos \frac{n \pi x}{\pi} \, dx \\
\text{อินเทグラทที่แต่งตัวของสังเคราะห์ จะได้}
\int_{0}^{2 \pi} x^2 \cos \frac{n \pi x}{\pi} \, dx &= x^2 \left( \frac{\sin \frac{n \pi x}{\pi}}{\frac{n \pi}{\pi}} \right) \bigg|_{0}^{2 \pi} - \int_{0}^{2 \pi} \left( \frac{\sin \frac{n \pi x}{\pi}}{\frac{n \pi}{\pi}} \right) 2x \, dx \\
&= \frac{1}{n \pi} \left( \sin \frac{2 \pi n}{\pi} \right) - \frac{2}{n \pi} \int_{0}^{2 \pi} x \sin \frac{n \pi x}{\pi} \, dx \\
&= 0 - \frac{2}{n \pi} \left[ x \left( \frac{-\cos \frac{n \pi x}{\pi}}{\frac{n \pi}{\pi}} \right) \bigg|_{0}^{2 \pi} - \int_{0}^{2 \pi} \left( \frac{-\cos \frac{n \pi x}{\pi}}{\frac{n \pi}{\pi}} \right) \, dx \right] \\
&= -\frac{2}{n \pi} \left[ \left\{ \frac{-2}{n \pi} \cos 2n \pi \right\} + \frac{1}{n^3 \pi^3} \int_{0}^{2 \pi} x \sin n \pi x \, dx \right] \\
&= \frac{4}{n^3 \pi^3} \left(1\right) - \frac{2}{n^3 \pi^3} \cdot \sin 2n \pi - 0 \\
&= \frac{4}{n^3 \pi^3} ; \quad \sin 2n \pi = 0
\end{align*}

หาค่า $b_n$ จากสูตร

\begin{align*}
b_n &= \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n \pi x}{\ell} \, dx \\
&= \frac{1}{1} \int_{0}^{2 \pi} \left( x^2 \right) \sin \frac{n \pi x}{\pi} \, dx \\
\text{อินเทグラทที่แต่งตัวของสังเคราะห์ จะได้}
b_n &= x^2 \left( \frac{-\cos \frac{n \pi x}{\pi}}{\frac{n \pi}{\pi}} \right) \bigg|_{0}^{2 \pi} - \int_{0}^{2 \pi} \left( \frac{-\cos \frac{n \pi x}{\pi}}{\frac{n \pi}{\pi}} \right) 2x \, dx \\
&= -\frac{4}{n \pi} \cos 2n \pi + \frac{2}{n \pi} \int_{0}^{2 \pi} x \cos \frac{n \pi x}{\pi} \, dx \\
&= -\frac{4}{n \pi} \left(1\right) + \frac{2}{n \pi} \left[ x \left( \frac{\sin \frac{n \pi x}{\pi}}{\frac{n \pi}{\pi}} \right) \bigg|_{0}^{2 \pi} - \int_{0}^{2 \pi} \left( \frac{\sin \frac{n \pi x}{\pi}}{\frac{n \pi}{\pi}} \right) \, dx \right]
\end{align*}
\[ f(x) = -\frac{4}{\pi^2} + \frac{2}{\pi^2} \left[ \frac{1}{\pi^2} \left| \sin 2\pi x - 0 \right| + \frac{1}{\pi^2} \cos \pi x \right]^2 \]

\[ = -\frac{4}{\pi^2} + \frac{2}{\pi^2} \left[ 0 + \frac{1}{\pi^2} \right] \cos 2\pi x = 1 \]

แทนค่า \( a_0, a_n \) และ \( b_n \) ลงในสูตรอนุกรมฟูเรียร์

\[ f(x) = \frac{1}{2} \left( \frac{8}{3} \right) + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[ \frac{4}{\pi^2} \cos \pi x - \frac{4}{\pi^2} \sin \pi x \right] \]

\[ = \frac{4}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos \pi x}{n^2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin \pi x}{n} \]

4. \( f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 < x < 2 \quad T = 2 \end{cases} \)

วิธีทำ เพราะว่า 1 คาบ = 2T = 2 เพราะฉะนั้น \( T = 1 \)

สูตรอนุกรมฟูเรียร์คือ

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right) \]

หาค่า \( a_0 \) จากสูตร

\[ a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \, dx \]

เลือก \( c = 0 \) จะได้

\[ a_0 = \frac{1}{2} \int_{-1}^{0} f(x) \, dx \]

\[ + \int_{1}^{2} (1) \, dx \]

\[ = 0 + \int_{1}^{2} 1 \, dx \]

\[ = 1 \]

สุภานิช นิยมศรี
หาค่า $a_n$ จากสูตร

$$a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \left[ \int_{0}^{0+2(1)} f(x) \cos \frac{n\pi x}{l} \, dx \right]$$

$$= \int_{0}^{1} (0) \cos n\pi x \, dx + \int_{1}^{2} (1) \cos n\pi x \, dx$$

$$= \frac{\sin n\pi x}{n\pi} \bigg|_{1}^{2}$$

$$= \frac{1}{n\pi} \left[ \sin 2n\pi - \sin n\pi \right]$$

$$= \frac{1}{n\pi} \left[ 0 \right] ; \quad \sin 2n\pi = \sin n\pi = 0$$

$$= 0$$

หาค่า $b_n$ จากสูตร

$$b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \left[ \int_{0}^{0+2(1)} f(x) \sin n\pi x \, dx \right]$$

$$= \int_{0}^{1} (0) \sin n\pi x \, dx + \int_{1}^{2} (1) \sin n\pi x \, dx$$

$$= 0 + \left( - \frac{\cos n\pi x}{n\pi} \right) \bigg|_{1}^{2}$$

$$= \frac{-1}{n\pi} \left[ \cos 2n\pi - \cos n\pi \right]$$

$$= \frac{-1}{n\pi} \left[ 1 - (-1)^n \right]$$

แทนค่า $a_0$, $a_n$ และ $b_n$ ลงในสูตรอนุกรมฟูเรียร์

$$f(x) = \frac{1}{2} (1) + \sum_{n=1}^{\infty} \left( 0 \cos n\pi x - \frac{1}{n\pi} \left[ 1 - (-1)^n \right] \sin n\pi x \right)$$

$$= \frac{1}{2} - \frac{1}{\pi} \left[ \frac{2}{1} \sin \pi x + 0 + \frac{2}{3} \sin 3\pi x + 0 + \frac{2}{5} \sin 5\pi x + \ldots \right]$$
\[ = \frac{1}{2} - \frac{2}{\pi} \left[ \sin \pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x + \ldots \right] \]
\[ = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin (2n-1)x}{2n-1} \]

5. \[ f(x) = \begin{cases} 0 & ; \quad 0 < x < 1 \\ x - 1 & ; \quad 1 < x < 2 ; \quad T = 2 \end{cases} \]

วิธีทำ  เพราะว่า 1 คาบ = 2\[ = 2 \] เพราะฉะนั้น \[ \ell = 1 \]

สูตรอนุกรมพีเรียกรูค

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \]

หาค่า \[ a_0 \] จากสูตร

\[ a_0 = \frac{1}{\ell} \int_{c}^{c+2\ell} f(x) \, dx \]

เลือก \[ c = 0 \]

\[ a_0 = \frac{1}{1} \int_{0}^{2\ell} f(x) \, dx \]

\[ = \int_{0}^{1} (0) \, dx + \int_{1}^{2} (x - 1) \, dx \]

\[ = 0 + \left[ \frac{x^2}{2} \right]_{1}^{2} \]

\[ = \frac{3}{2} - 1 \]

\[ = \frac{1}{2} \]

หาค่า \[ a_n \] จากสูตร

\[ a_n = \frac{1}{\ell} \int_{c}^{c+2\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx \]
\[ \frac{1}{n\pi} \left( 2 \sin 2n\pi - \sin n\pi \right) + \frac{1}{n^2\pi^2} \cos n\pi \]

\[ - \frac{1}{n\pi} \{ \sin 2n\pi - \sin n\pi \} \]

\[ = \frac{1}{n\pi} (0) + \frac{1}{n^2\pi^2} \{ \cos 2n\pi - \cos n\pi \} - \frac{1}{n\pi} (0) \]

\[ = \frac{1}{n^2\pi^2} \left[ 1 - (-1)^n \right] \]

พายค่า \( b_n \) จากสูตร

\[ b_n = \frac{1}{l} \int_{-l}^{+l} f(x) \sin \frac{n\pi x}{l} \, dx \]

\[ = \frac{1}{l} \int_{0}^{+2l} f(x) \sin \frac{n\pi x}{l} \, dx \]

\[ = \frac{1}{l} \left[ 0 \right]^{+2l} f(x) \sin \frac{n\pi x}{l} \, dx \]

\[ = \frac{1}{l} \int_{0}^{+l} (0) \sin \frac{n\pi x}{l} \, dx + \frac{1}{l} \int_{l}^{+2l} (0 - 1) \sin \frac{n\pi x}{l} \, dx \]

\[ = \frac{1}{l^2} \left[ 0 \right]^{+l} \sin \frac{n\pi x}{l} \, dx + \frac{1}{l^2} \left[ 0 \right]^{+2l} \sin \frac{n\pi x}{l} \, dx \]

\[ = \left( -\frac{1}{n\pi} \right) \int_{0}^{l} \left( -\frac{\cos n\pi x}{n\pi} \right) \, dx + \frac{1}{n\pi} \int_{l}^{2l} \cos n\pi \, dx \]

\[ = \frac{1}{n\pi} \left[ 2 \cos 2n\pi - \cos n\pi \right] + \frac{1}{n\pi} \int_{l}^{2l} \cos n\pi \, dx \]

\[ = \frac{1}{n\pi} \left[ \cos 2n\pi - \cos n\pi \right] + \frac{1}{n\pi} \left[ \sin 2n\pi - \sin n\pi \right] + \frac{1}{n\pi} \left[ (1 - (-1)^n) \right] \]

\[ = \frac{1}{n\pi} \left[ \cos 2n\pi - \cos n\pi \right] + \frac{1}{n\pi} \left[ \sin 2n\pi - \sin n\pi \right] + \frac{1}{n\pi} \left[ (1 - (-1)^n) \right] \]

\[ = \frac{1}{n\pi} \left[ \cos 2n\pi - \cos n\pi \right] + \frac{1}{n\pi} \left[ \sin 2n\pi - \sin n\pi \right] + \frac{1}{n\pi} \left[ (1 - (-1)^n) \right] \]

\[ = \frac{1}{n\pi} \left[ \cos 2n\pi - \cos n\pi \right] + \frac{1}{n\pi} \left[ \sin 2n\pi - \sin n\pi \right] + \frac{1}{n\pi} \left[ (1 - (-1)^n) \right] \]

\[ = \frac{1}{n\pi} \left[ \cos 2n\pi - \cos n\pi \right] + \frac{1}{n\pi} \left[ \sin 2n\pi - \sin n\pi \right] + \frac{1}{n\pi} \left[ (1 - (-1)^n) \right] \]

\[ = \frac{1}{n\pi} \left[ \cos 2n\pi - \cos n\pi \right] + \frac{1}{n\pi} \left[ \sin 2n\pi - \sin n\pi \right] + \frac{1}{n\pi} \left[ (1 - (-1)^n) \right] \]
แทนค่า $a_0$, $a_n$ และ $b_n$ ในสูตรอนุกรมฟูรีเย่ร์

$$f(x) = \frac{1}{2} \left( \frac{1}{2} \right) + \sum_{n=1}^{\infty} \left[ \frac{1 - (-1)^n}{n\pi} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right]$$

6. $f(x) = \begin{cases} x & ; \quad 0 < x < 1 \\ 1 & ; \quad 1 < x < 2 \quad , \quad T = 2 \end{cases}$

วิธีทำ

เพราะว่า $T = 2 \pi = 2$ เพราะฉะนั้น $f = 1$

สูตรอนุกรมฟูรีเย่ร์ คือ

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right)$$

หาค่า $a_0$ จากสูตร

$$a_0 = \frac{1}{T} \int_{c}^{c+2T} f(x) \, dx$$

เลือกค่า $c = 0$ ดังนั้น

$$a_0 = \frac{1}{1} \int_{0}^{1} f(x) \, dx$$

$$= \int_{0}^{1} x \, dx + \int_{1}^{2} (1) \, dx$$

$$= \frac{x^2}{2} \bigg|_0^1 + x \bigg|_1^2$$

$$= \frac{1}{2} + 1$$

$$= \frac{3}{2}$$

หาค่า $a_n$ จากสูตร

$$a_n = \frac{1}{T} \int_{c}^{c+2T} f(x) \cos \frac{n\pi x}{T} \, dx$$

$$= \frac{1}{1} \int_{0}^{1} f(x) \cos n\pi x \, dx$$

$$= \int_{0}^{1} (x \cos n\pi x) \, dx + \int_{1}^{2} (1) \cos n\pi x \, dx$$

$$= x \left( \frac{\sin n\pi x}{n\pi} \right) \bigg|_0^1 - \int_{0}^{1} \left( \frac{\sin n\pi x}{n\pi} \right) \, dx + \sin n\pi x \bigg|_1^2$$

มา 446 (H)
\[ b_n = \frac{1}{\ell} \int_{c}^{c+2\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx \]

\[ = \frac{1}{\ell} \int_{0}^{0+2(1)} f(x) \sin n\pi x \, dx \]

\[ = \int_{0}^{1} x \sin n\pi x \, dx + \int_{1}^{2} (1) \sin n\pi x \, dx \]

\[ = \frac{1}{n\pi} \left[ \cos n\pi x \right]_{0}^{1} - \int_{0}^{1} \left( \frac{-\cos n\pi x}{n\pi} \right) \, dx - \frac{\cos n\pi x}{n\pi} \bigg|_{1}^{2} \]

\[ = \frac{-1}{n\pi} \left( \cos n\pi - 0 \right) + \frac{\sin n\pi x}{n^2\pi^2} \bigg|_{0}^{1} - \frac{1}{n\pi} \left( \cos 2n\pi - \cos n\pi \right) \]

\[ = \frac{-(-1)^n}{n\pi} + \frac{1}{n^2\pi^2} \left( \sin n\pi - 0 \right) - \frac{1}{n\pi} \left\{ 1 - (-1)^n \right\} \]

\[ = \frac{-(-1)^n}{n\pi} + 0 - \frac{1}{n\pi} + \frac{(-1)^n}{n\pi} \]

\[ = \frac{-1}{n\pi} \]

แทนค่า \( a_0, a_n \) และ \( b_n \) ลงในสูตรอนุกรมฟูเรียร์

\[ f(x) = \frac{1}{2} \left( \frac{3}{2} \right) + \sum_{n=1}^{\infty} \left[ \frac{(-1)^n}{n^2\pi^2} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right] \]

\[ = \frac{3}{4} + \sum_{n=1}^{\infty} \left[ \frac{(-1)^n}{n^2\pi^2} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right] \]

\[ = \frac{3}{4} + \frac{1}{\pi} \left[ \frac{-2}{1} \cos \pi x + 0 - \frac{2}{3^2} \cos 3\pi x + 0 - \frac{2}{5^2} \cos 5\pi x + ... \right] \]

\[ - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n} \]

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KA 446 (ย)
\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi x)}{(2n-1)^2} &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n} \\
3 &= 2 - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x}{(2n-1)^2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n}
\end{align*}
\]

7. \( f(x) = \begin{cases} \frac{1}{2} - x & ; \ 0 < x < \frac{1}{2} \\ x - \frac{1}{4} & ; \ \frac{1}{2} < x < 1 \end{cases} \quad T = 1 \)

วิธีทำ เพราะว่า \( T = 2 \) เท่านั้น \( \ell = \frac{1}{2} \)

พิจารณาทั้งหมดที่เรียกได้

\( f(x) = -\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \)

หาค่า \( a_0 \) จากสูตร

\[ a_0 = \frac{1}{\ell} \int_{c}^{0+2\ell} f(x) \, dx \]

เลือกค่า \( c = 0 \) พิจารณา

\[ a_0 = \frac{1}{\frac{1}{2}} \int_{0}^{0+2\left(\frac{1}{2}\right)} f(x) \, dx \]

\[ = 2 \int_{0}^{1} f(x) \, dx \]

\[ = 2 \left[ \int_{0}^{1/2} x^2 \, dx + \int_{1/2}^{1} \left( x - \frac{3}{4} \right) \, dx \right] \]

\[ = 2 \left[ \frac{1}{2} \left( \frac{1}{2} \right)^2 + \frac{1}{2} \left( 1 - \frac{3}{4} \right) \right] \]

\[ = 0 \]

\( \text{MA 446 (H)} \)

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\[ a_n = \frac{1}{\ell} \int_0^{\ell+2\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx \]

\[ = \frac{1}{1} \int_0^{1/2} f(x) \cos 2n\pi x \, dx \]

\[ = 2 \int_0^{1/2} f(x) \cos 2n\pi x \, dx \]

\[ = 2 \left[ \int_0^{1/2} \left( \frac{1}{4} - x \right) \cos 2n\pi x \, dx + \int_0^{1/2} \left( x - \frac{3}{4} \right) \cos 2n\pi x \, dx \right] \]

\[ = 2 \left[ \frac{1}{4} \int_0^{1/2} \cos 2n\pi x \, dx - \int_0^{1/2} x \cos 2n\pi x \, dx + \int_0^{1/2} x \cos 2n\pi x \, dx - \frac{3}{4} \int_0^{1/2} \cos 2n\pi x \, dx \right] \]

พิจารณา

\[ \int_0^{1/2} \cos 2n\pi x \, dx = \frac{\sin 2n\pi x}{2n\pi} \bigg|_0^{1/2} \]

\[ = \frac{1}{2n\pi} \left( \sin n\pi - 0 \right) \]

\[ = 0 \]

\[ \int_0^{1/2} x \cos 2n\pi x \, dx = x \left( \frac{\sin 2n\pi x}{2n\pi} \right) \bigg|_0^{1/2} - \int_0^{1/2} \left( \frac{\sin 2n\pi x}{2n\pi} \right) \, dx \]

\[ = \frac{1}{2n\pi} \left( \frac{1}{2} \sin n\pi - 0 \right) = \frac{1}{4n^2\pi^2} \]

\[ = 0 + \frac{1}{4n^2\pi^2} \cos n\pi - 1 \]

\[ = (-1)^n \frac{1}{4n^2\pi^2} \]

\[ \int_0^{1/2} x \cos 2n\pi x \, dx = x \left( \frac{\sin 2n\pi x}{2n\pi} \right) \bigg|_0^{1/2} - \int_0^{1/2} \left( \frac{\sin 2n\pi x}{2n\pi} \right) \, dx \]

\[ = \frac{1}{2n\pi} \left( \sin 2n\pi - \frac{1}{2} \sin n\pi \right) = \frac{1}{4n^2\pi^2} \cos 2n\pi x \bigg|_{1/2}^{1/2} \]
\[a_n = 2 \left[ \frac{1}{4} (0) - \frac{(1)^n - 1}{4n^2 \pi^2} \right] + \frac{1 - (1)^n}{4n^2 \pi^2} \left( \frac{3}{4} \right) \]

\[b_n = \frac{1}{\ell} \int_{0}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx\]

\[= \frac{1}{\ell} \int_{0}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx\]

\[= 2 \int_{0}^{1} f(x) \sin 2n\pi x \, dx\]

\[= 2 \left[ \frac{1}{4} \left( \frac{1}{4} - x \right) \sin 2n\pi x \, dx + \int_{1/2}^{1} \left( x - \frac{3}{4} \right) \sin 2n\pi x \, dx \right]\]

\[= 2 \left[ \frac{1}{4} \int_{0}^{1/2} \sin 2n\pi x \, dx - \int_{0}^{1/2} x \sin 2n\pi x \, dx + \int_{1/2}^{1} \sin 2n\pi x \, dx \right]\]

\[= \frac{3}{4} \int_{0}^{1/2} \sin 2n\pi x \, dx\]

\[\text{Ma 446 (H)}\]
\[\int_0^{1/2} \sin 2\pi x \, dx = \left. -\cos 2\pi x \right|_0^{1/2} \]
\[= -\frac{1}{2\pi} \left\{ \cos n\pi - 1 \right\} \]
\[= \frac{1 - (-1)^n}{2n} \]  \hspace{1cm} \ldots \ldots (7)

\[\int_0^{1/2} \cos 2\pi x \, dx = \left. \left( -\frac{\cos 2\pi x}{2\pi} \right) \right|_0^{1/2} - \frac{1}{2\pi} \int_0^{1/2} \left( -\cos 2\pi x \right) \, dx \]
\[= \frac{-1}{2\pi} \left\{ \frac{1}{2} \cos n\pi \right\} + \frac{\sin 2\pi x}{4n^2\pi^2} \left|_0^{1/2} \right. \]
\[= \frac{-(-1)^n}{4n} \]  \hspace{1cm} \ldots \ldots (8)

\[\int_0^{1} \cos 2\pi x \, dx = \left. \left( -\frac{\cos 2\pi x}{2\pi} \right) \right|_0^{1} - \frac{1}{2\pi} \int_0^{1} \left( -\cos 2\pi x \right) \, dx \]
\[= \frac{-1}{2\pi} \left\{ \cos 2\pi - \cos n\pi \right\} \]
\[= \left\{ \frac{1}{2} \sin \frac{2\pi n}{2\pi} \right\} + \frac{1}{4n^2\pi^2} \left\{ \sin 2\pi - \sin n\pi \right\} \]
\[= \frac{-1}{2\pi} \left\{ \frac{1}{2} \sin \frac{2\pi n}{2\pi} \right\} \]  \hspace{1cm} \ldots \ldots (9)

\[\int_0^{1} \sin 2\pi x \, dx = \left. -\cos 2\pi x \right|_0^{1/2} \]
\[= \frac{-1}{2\pi} \left\{ \cos 2\pi - \cos n\pi \right\} \]
\[= \frac{-1}{2\pi} \left\{ 1 - (-1)^n \right\} \]  \hspace{1cm} \ldots \ldots (10)

แทนค่า (7), (8), (9) และ (10) ใน (6) จะได้

\[b_n = 2 \left[ \frac{1}{4} \left\{ 1 - (-1)^n \right\} - \frac{1}{4n\pi} \right] - \frac{-(-1)^n}{4n\pi} - \frac{2 - (-1)^n}{4n\pi} + \frac{3}{4} \left\{ \frac{1}{2n\pi} \right\} \]
\[ f(x) = \frac{1}{2} (0) + \sum_{n=1}^{\infty} \left( \frac{1 - (-1)^n}{n^2 \pi^2} \right) \cos 2n\pi x + (0) \sin 2n\pi x \]

\[ = \frac{1}{\pi} \left[ \frac{2}{1^2} \cos 2\pi x + 0 + \frac{2}{3^2} \cos 6\pi x + 0 + \frac{2}{5^2} \cos 10\pi x + \ldots \right] \]

\[ = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos 2(2n-1)\pi x}{(2n-1)^2} \]

8. \[ f(x) = \begin{cases} 
8 & ; \quad 0 < x < 2 \\
-8 & ; \quad 2 < x < 4 ; \quad \Gamma = 4 
\end{cases} \]

วิธีทำ

ที่จุด \( x = 0 \) กระบุ \( f = 2 \)

\[ \sum_{n=1}^{\infty} \frac{\cos 2(2n-1)\pi x}{(2n-1)^2} \]

หา \( a_0 \) จากสูตร

\[ a_0 = \frac{1}{\Gamma} \int_{c}^{c+2\Gamma} f(x) \, dx \]

เมื่อ \( c = 0 \) ดังนั้น

\[ a_0 = \frac{1}{2} \int_{0}^{2} f(x) \, dx \]

\[ = \frac{1}{2} \left[ \int_{0}^{2} (8) \, dx + \int_{2}^{4} (-8) \, dx \right] \]

\[ = \frac{1}{2} \left[ 8x \bigg|_{0}^{2} + 8x \bigg|_{2}^{4} \right] \]

\[ = 4 \left[ 2 - 2 \right] \]

\[ = 0 \]
หาค่า $a_n$ จากสูตร

$$a_n = \frac{1}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx$$

$$= \frac{1}{2} \int_0^{\ell} f(x) \cos \frac{n\pi x}{2} \, dx$$

$$= \frac{1}{2} \left[ \int_0^{\ell} (8) \cos \frac{n\pi x}{2} \, dx + \int_0^{\ell} (-8) \cos \frac{3n\pi x}{2} \, dx \right]$$

$$= \frac{1}{2} \left[ 8 \left( \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) \bigg|_0^\ell - 8 \left( \frac{\sin \frac{3n\pi x}{2}}{\frac{3n\pi}{2}} \right) \bigg|_0^\ell \right]$$

$$= 4 \left[ \frac{2}{n\pi} \sin n\pi - \frac{2}{n\pi} \{ \sin 2n\pi - \sin n\pi \} \right]$$

เพราะว่า $\sin 2n\pi = \sin n\pi = 0$ ดังนั้น

$$a_n = 4 \Rightarrow 0 = 0$$

หาค่า $b_n$ จากสูตร

$$b_n = \frac{1}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx$$

$$= \frac{1}{2} \int_0^{\ell} f(x) \sin \frac{n\pi x}{2} \, dx$$

$$= \frac{1}{2} \left[ \int_0^{\ell} (8) \sin \frac{n\pi x}{2} \, dx + \int_0^{\ell} (-8) \sin \frac{3n\pi x}{2} \, dx \right]$$

$$= 4 \left[ \frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \bigg|_0^\ell + \frac{\cos \frac{3n\pi x}{2}}{\frac{3n\pi}{2}} \bigg|_0^\ell \right]$$

$$= 4 \left[ \frac{2}{n\pi} \{ \cos n\pi - 1 \} + \frac{2}{n\pi} \{ \cos 2n\pi - \cos n\pi \} \right]$$

$$= \frac{-8}{n\pi} \left\{ (-1)^n - 1 - 1 + (-1)^n \right\}$$

$$= \frac{16}{n\pi} \left[ 1 - (-1)^n \right]$$
แทนค่า $a_0$, $a_n$ และ $b_n$ ลงในสูตรอนุกรมฟูเรียร์

$$f(x) = \frac{1}{2} \left[ a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \right]$$

$$f(x) = \frac{16}{\pi} \left[ \frac{2}{1} \sin \frac{\pi x}{2} + 0 + \frac{2}{3} \sin \frac{3\pi x}{2} + 0 + \frac{2}{5} \sin \frac{5\pi x}{2} + \ldots \right]$$

$$= \frac{32}{\pi} \left[ \sin \frac{\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} + \frac{1}{5} \sin \frac{5\pi x}{2} + \ldots \right]$$

$$= \frac{32}{\pi} \sum_{n=1}^{\infty} \sin \frac{2n-1}{2} \frac{\pi x}{\ell}$$

9. $f(x) = 4x ; 0 < x < 10$

วิธีทำ สูตรอนุกรมฟูเรียร์คือ

$$f(x) = \frac{1}{\ell} \left[ a_0 + \sum a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right]$$

เพื่อรู้ว่า 1 ลบ $2\ell = 10$ เพราะฉนั้น $\ell = 5$ หาค่า $a_0$ จากสูตร

$$a_0 = \frac{1}{\ell} \int_{c}^{c+2\ell} f(x) \, dx$$

เลือก $c = 0$ ดังนี้

$$\int_{0}^{0+2(5)} (4x) \, dx$$

$$= \frac{1}{5} \int_{0}^{2(5)} (4x) \, dx$$

$$= \frac{1}{5} (2x^2) \bigg|_{0}^{10}$$

$$= \frac{2}{5} (100) = 40$$

หาค่า $a_n$ จากสูตร

$$a_n = \frac{1}{\ell} \int_{c}^{c+2\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx$$

$$= \frac{1}{5} \int_{0}^{0+2(5)} (4x) \cos \frac{n\pi x}{5} \, dx$$

$$= \frac{4}{5} \int_{0}^{10} x \cos \frac{n\pi x}{5} \, dx$$
\[
\begin{align*}
&= 4 \left[ x \left( \frac{\sin \frac{\pi x}{5}}{\sin \frac{\pi}{5}} \right) \right]_{0}^{10} - \left[ \frac{\sin \frac{\pi x}{5}}{\sin \frac{\pi}{5}} \right]_{0}^{10} \\
&= 4 \left[ \frac{5}{\pi} \right]_{0}^{10} \sin 2\pi - 0 + \frac{25}{\pi^2} \cos \left( \frac{\pi x}{5} \right)_{0}^{10} \\
&= 4 \left[ \frac{5}{\pi} \right]_{0}^{10} \sin 2\pi - 0 + \frac{25}{\pi^2} \cos 2\pi - 1 \\
&= \frac{20}{\pi^2} \left[ 1 - 1 \right] \\
&= 0 \\
\end{align*}
\]

หาค่า \(b_n\) จากทฤษฎี:

\[
\begin{align*}
\frac{1}{\ell} \int_{a}^{b} f(x) \sin \frac{\pi x}{\ell} \, dx \\
= \frac{1}{\ell} \left[ \int_{0}^{2\pi} (4x) \sin \frac{\pi x}{\ell} \, dx \right] \\
= \frac{4}{\ell} \left[ \int_{0}^{10} x \sin \frac{\pi x}{5} \, dx \right] \\
&= \frac{4}{\ell} \left[ x \left( \frac{\sin \frac{\pi x}{5}}{\sin \frac{\pi}{5}} \right) \right]_{0}^{10} - \left[ \frac{\sin \frac{\pi x}{5}}{\sin \frac{\pi}{5}} \right]_{0}^{10} \\
&= \frac{4}{\ell} \left[ \frac{5}{\pi} \right]_{0}^{10} \cos 2\pi - 0 + \frac{25}{\pi^2} \sin \left( \frac{\pi x}{5} \right)_{0}^{10} \\
&= \frac{4}{\ell} \left[ \frac{5}{\pi} \right]_{0}^{10} \cos 2\pi - 0 + \frac{25}{\pi^2} \sin 2\pi - 0 \\
&= \frac{4}{\ell} \left[ \frac{50}{\pi} + 0 \right] \text{; sin } 2\pi = 0 \\
&= \frac{-40}{\pi} \\
\end{align*}
\]
แทนค่า $a_n$, $b_n$ และ $b_n$ ลงในสูตรอนุกรมฟูริเยร์

$$f(x) = \frac{1}{2} (40) + \sum_{n=1}^{\infty} \left[ (0) \cos \frac{n \pi x}{5} + \left( -\frac{40}{n \pi} \right) \sin \frac{n \pi x}{5} \right]$$

$$= 20 - \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n \pi x}{5}$$

10. จงตรวจสอบอนุกรมฟูริเยร์ของฟังก์ชันมีค่า ซึ่งนิยามเป็น

$$f(x) = \begin{cases} 
\cos x & ; -\pi < x < 0 \\
\sin x & ; 0 < x < \pi 
\end{cases}$$

วิธีทำ สูตรอนุกรมฟูริเยร์ คือ

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

หาค่า $a_0$ จากสูตร

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$= \frac{1}{\pi} \left[ \int_{0}^{\pi} \cos x \, dx + \int_{-\pi}^{0} \sin x \, dx \right]$$

$$= \frac{1}{\pi} \left[ \sin x \bigg|_{0}^{\pi} + (\cos x) \bigg|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[ 0 - (1 - 1) \right]$$

$$= \frac{1}{\pi} \left[ 0 - \{1 - 1\} \right]$$

$$= \frac{0}{\pi}$$

หาค่า $a_n$ จากสูตร

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[ \int_{0}^{\pi} \cos x \cos nx \, dx + \int_{-\pi}^{0} \sin x \cos nx \, dx \right]$$
\[
\begin{align*}
&= \frac{1}{\pi}\left[\frac{1}{2} \int_{-\pi}^{\pi} \cos(l+n)x + \cos(l-n)x \, dx \right] \\
&= \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} \left( \frac{\sin(1+n)x}{1 + n} + \frac{\sin(1-n)x}{1 - n} \right) \, dx \right] \\
&\quad + \left[ \frac{\cos(1+n)x}{1 + n} - \frac{\cos(1-n)x}{1 - n} \right] \bigg|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} \frac{0 - \sin(1+n)(-\pi)}{1 + n} + \frac{0 - \sin(1-n)(-\pi)}{1 - n} \right] \\
&\quad - \left\{ \frac{\cos((1+n)\pi - 1)}{1 + n} - \frac{\cos((1-n)\pi - 1)}{1 - n} \right\} \\
\end{align*}
\]

พิจารณา
\begin{align*}
\sin((1+n)\pi) &= \sin((\pi+n\pi)) = -\sin n\pi = 0 \\
\sin((1-n)\pi) &= \sin((\pi-n\pi)) = \sin n\pi = 0 \\
\cos((1+n)\pi) &= \cos((\pi+n\pi)) = -\cos n\pi = (-1)^n \\
\csc((1-n)\pi) &= \csc((\pi-n\pi)) = -\csc n\pi = (-1)^n \\
\end{align*}

พิจารณา
\begin{align*}
a_n &= \frac{1}{2\pi} \left[ 0 + 0 - \left\{ \frac{(-1)^n - 1}{1 + n} - \frac{(-1)^n - 1}{1 - n} \right\} \right] \\
&= \frac{1}{2\pi} \left[ 1 + (-1)^n \left\{ \frac{1}{1 + n} + \frac{1}{1 - n} \right\} \right] \\
&= \frac{1}{2\pi} \left[ \frac{1 + (-1)^n}{1 - n^2} \right] \\
&= \frac{1}{\pi(1 - n^2)} \quad ; n \neq 1
\end{align*}

หาค่า \( a_1 \) ใหม่ จากค่า \( a_n \) โดยแทนค่า \( n = 1 \) ดังนี้
\begin{align*}
a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx
\end{align*}
\[ \begin{align*}
&= \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} \cos x \cos x \, dx + \int_{0}^{\pi} \sin x \cos x \, dx \right] \\
&= \frac{1}{\pi} \left[ \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2x) \, dx + \frac{1}{2} \int_{0}^{\pi} \sin 2x \, dx \right] \\
&= \frac{1}{2\pi} \left[ \left( x + \frac{\sin 2x}{2} \right) \bigg|_{-\pi}^{\pi} - \cos \frac{2x}{2} \bigg|_{0}^{\pi} \right] \\
&= \frac{1}{2\pi} \left[ 0 - (-\pi) \right] + \frac{1}{2} \left[ 0 - \sin (-2\pi) \right] - \frac{1}{2} \left[ \cos 2\pi - 1 \right] \\
&= \frac{1}{2\pi} \left[ (\pi) + \frac{1}{2} \left[ 0 \right] - \frac{1}{2} \left[ 1 \right] \right] \\
&= \frac{1}{2}
\end{align*} \]

Hence

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \]

\[ b_n = \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} \cos \sin x \, dx + \int_{0}^{\pi} \sin x \sin nx \, dx \right] \\
= \frac{1}{\pi} \left[ \frac{1}{2} \int_{-\pi}^{\pi} \left( \sin (1 + n)x - \sin (1 - n)x \right) \, dx \right] \\
+ \frac{1}{2} \left[ \int_{0}^{\pi} \left( \cos (1 - n)x - \cos (1 + n)x \right) \, dx \right] \\
= \frac{1}{2\pi} \left[ \left( \frac{-\cos (1 + n)x + \cos (1 - n)x}{1 + n} \right) \bigg|_{0}^{\pi} \right] \\
+ \left( \sin (1 - n)x \bigg|_{1 - n}^{1 + n} \right) \bigg|_{0}^{\pi} \\
= \frac{1}{2\pi} \left[ \left( 1 - \cos (1 + n)\pi \right) + \frac{1 - \cos (1 - n)\pi}{1 + n} \right] \\
+ \left( \sin (1 - n)\pi - 0 \right) - \frac{\sin (1 + n)\pi - 0}{1 + n} \\
\]

Hence \( \cos (1 + n)\pi = -(-1)^n \) \( \cos (1 - n)\pi = -(-1)^n \)

\[ \text{and} \quad \sin (1 + n)\pi = 0 \quad \sin (1 - n)\pi = 0 \]
ตัวनั้น

\[ b_n = \frac{1}{2\pi} \left( \frac{\{ 1 + \frac{(-1)^n}{1 + n} \}}{1 + n} + \frac{\{ 1 + \frac{(-1)^n}{1 - n} \}}{1 - n} + 0 - 0 \right) \]

\[ = \frac{1}{2\pi} \left( \frac{1 + \frac{(-1)^n}{1 + n}}{1 + n} + \frac{1}{1 - n} \right) \]

\[ = \frac{1}{2\pi} \left( \frac{1 + \frac{(-1)^n}{1 + n}}{1 - n^2} \right) \]

\[ = \frac{n \cdot 1 + \frac{(-1)^n}{n}}{\pi(1 - n^2)} \quad : \quad n \neq 1 \]

หากค่า \( b_1 \) จากสูตร \( b_n \) โดยแทนค่า \( n = 1 \) เป็นงวดนี้

\[ b_1 = \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} f(x) \sin x \, dx \right] \]

\[ = \frac{1}{\pi} \left[ \int_{-\pi}^{0} \cos x \sin x \, dx + \int_{0}^{\pi} \sin x \sin x \, dx \right] \]

\[ = \frac{1}{\pi} \left[ \int_{-\pi}^{0} \sin 2x \, dx + \int_{0}^{\pi} 1 - \cos 2x \, dx \right] \]

\[ = \frac{1}{2\pi} \left[ -\cos 2x \bigg|_{-\pi}^{0} + \left( x - \sin 2x \right) \bigg|_{0}^{\pi} \right] \]

\[ = \frac{1}{2\pi} \left[ -\frac{1}{2} (1 - \cos (-2\pi)) + \pi - 0 \right] \]

\[ = \frac{1}{2\pi} \left[ -\frac{1}{2} (1 - 1) + \pi \right] \]

\[ = \frac{1}{2} \]

แทนค่า \( a_0, a_n \) และ \( b_n \) ในสูตรออนุกรมฟูเรียร์

\[ f(x) = \frac{1}{2} \left( \frac{2}{\pi} \right) + \left( \frac{1}{2} \right) \cos x + \left( \frac{1}{2} \right) \sin x + \sum_{n=2}^{\infty} \left[ \frac{1 + \frac{(-1)^n}{\pi(1 - n^2)}}{\pi(1 - n^2)} \right] \cos nx \]

\[ + n \cdot 1 + \frac{(-1)^n}{\pi(1 - n^2)} \sin nx \]
\[
I = \frac{1}{2} (\cos x + \sin x) - \sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n^2 - 1} \cos nx
\]

11. จงพิสูจน์ว่าสำหรับ \( 0 \leq x \leq \pi \)

(ก) \( x(\pi - x) = \frac{\pi^2}{6} - \left( \frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots \right) \)

(ข) \( x(\pi - x) = \frac{8}{\pi} \left( \frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots \right) \)

วิธีทำ (ก) สูตรอนุกรมสันทนาวิทยาที่คือ

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx \]

หาค่า \( a_0 \) จากสูตร

\[
a_0 = \frac{2}{\pi} \int_{0}^{\pi} f(x) \, dx
\]

\[
= \frac{2}{\pi} \int_{0}^{\pi} x(\pi - x) \, dx
\]

\[
= \frac{2}{\pi} \left[ \pi \int_{0}^{\pi} x \, dx - \int_{0}^{\pi} x^2 \, dx \right]
\]

\[
= \frac{2}{\pi} \left[ \pi \left( \frac{x^2}{2} \right) \bigg|_{0}^{\pi} - \frac{x^3}{3} \bigg|_{0}^{\pi} \right]
\]

\[
= \frac{2}{\pi} \left[ \frac{\pi^3}{2} - \frac{\pi^3}{3} \right]
\]

\[
= \frac{2}{\pi} \left[ \frac{\pi^3}{6} \right]
\]

\[
= \frac{\pi^3}{3}
\]

หาค่า \( a_n \) จากสูตร

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx
\]

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พิจารณา

\[
\int_0^\pi x \cos nx \, dx = \left[ -\frac{\sin nx}{n} \right]_0^\pi = \frac{\pi}{n} \left( \frac{\sin nx}{n} \right)_{\pi = 0} \]

\[
= \frac{1}{n} \left\{ \pi \sin n\pi - 0 \right\} + \frac{1}{n^2} \cos nx \right|_0^\pi
\]

\[
= 0 + \frac{1}{n^2} \left\{ \cos n\pi - 1 \right\}
\]

\[
= \frac{(-1)^n - 1}{n^2}
\]

\[
\int_0^\pi x^2 \cos nx \, dx = x^2 \left( \frac{\sin nx}{n} \right) \right|_{0}^{\pi} - \frac{\pi}{n} \left( \frac{\sin nx}{n} \right) 2x \, dx
\]

\[
= \frac{1}{n} \left\{ \pi^2 \sin n\pi - 0 \right\} - \frac{2}{n^2} \left[ -\frac{\pi}{n} \sin nx \right]_{0}^{\pi}
\]

\[
= 0 - \frac{2}{n} \left[ -\frac{1}{n} \left\{ \pi \cos n\pi - 0 \right\} + \frac{\sin nx}{n^2} \right|_0^\pi
\]

\[
= -\frac{2}{n} \left[ \frac{\pi (-1)^n}{n} + \frac{1}{n^2} \left\{ \sin n\pi - 0 \right\} \right]
\]

\[
= \frac{2\pi (-1)^n}{n^2}
\]

แทนค่า (3) ใน (1) จะได้

\[
a_n = \frac{2}{\pi} \left[ \pi \left\{ \frac{(-1)^n - 1}{n^2} \right\} - \frac{2\pi (-1)^n}{n^2} \right]
\]

\[
= \frac{2}{\pi} \left\{ \frac{(-1)^n}{n^2} - \frac{1}{n^2} - 2(-1)^n \right\}
\]

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แทนค่า $a_0, a_n$ ลงในสูตรอนุกรมฟูรีย์ ได้

$$f(x) = \frac{1}{2} \left( \frac{\pi^2}{3} \right) + \sum_{n=1}^{\infty} \frac{-2 \left[ 1 + (-1)^n \right]}{n^2} \cos nx$$

$$= \frac{\pi^2}{6} \cdot 2 \left[ 0 + \frac{2}{2^2} \cos 2x + 0 + \frac{2}{4^2} \cos 4x + 0 + \frac{2}{6^2} \cos 6x + \ldots \right]$$

$$= \frac{\pi^2}{6} - \frac{1}{1^2} \cos 2x + \frac{1}{2^2} \cos 4x + \frac{1}{3^2} \cos 6x + \ldots$$

$$= \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^2}$$

(2) สูตรอนุกรมฟูรีย์ คือ

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

หาค่า $b_n$ จากรูป

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[ \int_{0}^{\pi} x \sin nx \, dx - \int_{0}^{\pi} x^2 \sin nx \, dx \right] \ldots (4)$$

พิจารณา

$$\int_{0}^{\pi} x \sin nx \, dx = x \left( \frac{-\cos nx}{n} \right) \bigg|_{0}^{\pi} - \int_{0}^{\pi} \left( \frac{-\cos nx}{n} \right) \, dx$$

$$= -\frac{1}{n} \left[ \pi \cos n\pi \right] + \frac{\sin nx}{n^2} \bigg|_{0}^{\pi}$$

$$= \frac{\pi (1)^0}{n} + \frac{1}{n^2} \left[ \sin n\pi \right] \quad 0$$

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\[
\int_0^{\pi} x^2 \sin nx \, dx = x^2 \left( -\frac{\cos nx}{n} \right) \Big|_0^{\pi} - \int_0^{\pi} \left( -\frac{\cos nx}{n} \right) 2x \, dx 
\]

\[
= -\frac{1}{n} \pi^2 \cos n\pi - 0 + \frac{2}{n} \int_0^{\pi} x \cos nx \, dx 
\]

\[
= -\frac{\pi^2}{n}(-1)^n + \frac{2}{n} \left\{ x \left( -\frac{\sin nx}{n} \right) \Big|_0^{\pi} - \int_0^{\pi} \left( -\frac{\sin nx}{n} \right) \, dx \right\} 
\]

\[
= -\frac{\pi^2}{n}(-1)^n + \frac{2}{n} \left\{ 0 + \frac{1}{n^2} \cos n\pi \right\} 
\]

\[
= -\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3} \left\{ (-1)^n - 1 \right\} 
\]

แทนค่า (5) และ (6) ใน (4)

\[
b_n = \frac{2}{\pi} \left[ \pi \left\{ -\frac{\pi(-1)^n}{n} \right\} - \left\{ -\frac{\pi^2(-1)^n}{n} + \frac{2}{n^3} \left\{ (-1)^n - 1 \right\} \right\} \right] 
\]

\[
= \frac{2}{\pi} \left[ -\frac{n^2(-1)^n}{n} \pi^2 \left(-1\right)^n - \frac{2}{n^3} \left\{ (-1)^n - 1 \right\} \right] 
\]

\[
= \frac{4 \left\{ 1 - (-1)^n \right\}}{\pi n^3} 
\]

แทนค่า \( b_n \) ในสูตรอนุกรมฟูรีเยร์

\[
f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{\pi n^3} \sin \pi x 
\]

\[
= \frac{4}{\pi} \left[ \frac{2}{1^3} \sin x + \frac{2}{3^3} \sin 3x + \frac{2}{5^3} \sin 5x \right] 
\]

\[
= \frac{8}{\pi} \left[ \sin x \frac{1}{1^3} + \sin 3x \frac{1}{3^3} + \sin 5x \frac{1}{5^3} + \ldots \right] 
\]
\[ f(x) = \begin{cases} 
2 & ; \quad 0 < x < \frac{2\pi}{3} \\
\frac{2\pi}{3} & ; \quad \frac{2\pi}{3} < x < \frac{4\pi}{3} \\
0 & ; \quad \frac{4\pi}{3} < x < 2\pi
\end{cases} \]

วิธีทํา คูณความฟูเรียร์ คือ

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \]

หาค่า \(a_0\) จากสูตร

\[ a_0 = \frac{1}{\ell} \int_{c}^{c + 2\ell} f(x) \, dx \]

พิจารณา 1 ค่า = \(2\ell = 2\pi\) เพาะ\(\ell = \pi\)

เลือกแทนค่า \(c = 0\) ดังนี้

\[ a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \, dx \]

\[ = \frac{1}{\pi} \left[ \int_{0}^{\frac{2\pi}{3}} (2) \, dx + \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1) \, dx + \int_{\frac{4\pi}{3}}^{2\pi} (0) \, dx \right] \]

\[ = \frac{1}{\pi} \left[ 2x \Big|_{0}^{\frac{2\pi}{3}} + \left( \frac{4\pi}{3} + 0 \right) \right] \]

\[ = \frac{1}{\pi} \left[ 2 \left( \frac{2\pi}{3} \right) + \left( \frac{4\pi}{3} - \frac{2\pi}{3} \right) \right] \]

\[ = \frac{1}{\pi} \left[ \frac{4\pi}{3} + \frac{2\pi}{3} \right] \]

\[ = \frac{1}{\pi} (2\pi) \]

\[ = 2 \]

หาค่า \(a_n\) จากสูตร

\[ a_n = \frac{1}{\ell} \int_{c}^{c + 2\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx \]

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\[
\begin{align*}
\int_{0}^{2\pi} f(x) \cos nx \, dx &= \frac{1}{\pi} \left[ \int_{0}^{2\pi/3} 2 \cos nx \, dx + \int_{2\pi/3}^{4\pi/3} \cos nx \, dx + \int_{4\pi/3}^{2\pi} 0 \cos nx \, dx \right] \\
&= \frac{1}{\pi} \left[ \int_{0}^{\pi/3} \frac{2 \sin nx}{n} \, dx + \int_{\pi/3}^{2\pi/3} \frac{\sin nx}{n} \, dx \right] + \frac{1}{n} \left[ \sin \frac{4n\pi}{3} \sin \frac{2n\pi}{3} \right] \\
&= \frac{1}{\pi} \left[ \frac{2}{n} \left( \sin \frac{2n\pi}{3} \right) - 0 \right] + \frac{1}{n} \left[ \sin \frac{4n\pi}{3} \sin \frac{2n\pi}{3} \right] \\
&= \frac{1}{\pi} \left[ \frac{1}{n} \sin \frac{2n\pi}{3} + \frac{1}{n} \sin \frac{4n\pi}{3} \right] \\
&= \frac{1}{n\pi} \left( \sin \frac{2n\pi}{3} + \sin \frac{4n\pi}{3} \right)
\end{align*}
\]

หาค่า \( b_n \) จากสูตร

\[
\begin{align*}
b_n &= \frac{1}{\ell} \int_{c}^{c+2\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx \\
&= \frac{1}{\pi} \left[ \int_{0}^{2\pi} f(x) \sin nx \, dx \right] \\
&= \frac{1}{\pi} \left[ \int_{0}^{2\pi/3} 2 \sin nx \, dx + \int_{2\pi/3}^{4\pi/3} \sin nx \, dx + \int_{4\pi/3}^{2\pi} 0 \sin nx \, dx \right] \\
&= \frac{1}{\pi} \left[ -\frac{2 \cos nx}{n} \int_{0}^{\pi/3} - \frac{\cos nx}{n} \int_{\pi/3}^{2\pi/3} + 0 \right] \\
&= \frac{1}{\pi} \left[ -\frac{2}{n} \left( \cos \frac{2n\pi}{3} + 1 \right) - \frac{1}{n} \left( \cos \frac{4n\pi}{3} + \cos \frac{2n\pi}{3} \right) \right] \\
&= \frac{1}{\pi} \left[ \frac{2}{n} \cos \frac{2n\pi}{3} + \frac{2}{n} - \frac{1}{n} \cos \frac{4n\pi}{3} + \frac{1}{n} \cos \frac{2n\pi}{3} \right] \\
&= \frac{1}{\pi n} \left( \frac{2}{n} - \frac{1}{n} \cos \frac{2n\pi}{3} - \frac{1}{n} \cos \frac{4n\pi}{3} \right)
\end{align*}
\]
\[ f(x) = \frac{1}{2} (2) + \sum_{n=1}^{\infty} \left( \frac{1}{n\pi} \left( \sin \frac{2n\pi}{3} + \sin \frac{4n\pi}{3} \right) \cos nx \right) \\
\quad + \frac{1}{n\pi} \left( 2 - \cos \frac{2n\pi}{3} - \cos \frac{4n\pi}{3} \right) \sin nx \]

\[ = 1 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \sin \frac{2n\pi}{3} + \sin \frac{4n\pi}{3} \right) \cos nx \]

\[ + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( 2 - \cos \frac{2n\pi}{3} - \cos \frac{4n\pi}{3} \right) \sin nx \]
และประเภททัศนคติ 2.5

1. ใช้ทฤษฎีจากประเภททัศนคติ 2.1 ข้อ 2 จงแสดงว่า

(ก) \(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{6}\)

(ข) \(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots = \frac{\pi^2}{12}\)

(ค) \(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots = \frac{\pi^2}{8}\)

วิธีทำ อนุกรมที่เรียกว่าจากประเภททัศนคติ 2.1 ข้อ 2 คือ

\[f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx\]

(1)

ในเมื่อ \(f(x) = x^2; |x| < \pi\)

(ก) เพราะว่า \(f(x)\) มีทั้งในช่วง \((-\pi, \pi)\) เพราะฉะนั้น ถ้าท่านค่า \(x = \pi\) ซึ่งเป็นจุดไม่ต่อเนื่องใน (1) แล้วใช้เรื่องโอชีริคคด จะได้ค่าเข้าของสมการ

\[f(\pi) = \frac{f(\pi + 0) + f(\pi - 0)}{2} = \frac{\pi^2 + \pi^2}{2} = \pi^2\]

ตัวหนัง

\[\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx\]

\[\pi^2 - \frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \pi x\]

\[2\pi^2 = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \text{เพราะว่า } \frac{(-1)^n}{\pi^n} = 1\]

\[\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\]

หรือ \(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}\)

(ข) แทนค่า \(x = 0\) ลงใน (1) จะได้

\[f(0) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\]

(1)
แต่จุด $x = 0$ เป็นจุดต่อเนื่อง ดังนั้น $f'(0) = (0)^2 = 0$ นั่นคือ

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$- \frac{\pi^2}{3} = 4 \left( -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots \right)$$

$$- \frac{\pi^2}{12} = - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots$$

ถ้า $-1$ คูณผลลดสมการ จะได้

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots = \frac{\pi^2}{12}$$

(ค) ใช้ผลจากข้อ (ข) เพื่อว่า

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots = \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right) - \frac{1}{2^2} \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right)$$

นั่นคือ

$$\frac{\pi^2}{12} = \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right) - \frac{1}{2^2} \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right)$$

แต่

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{6}$$

ดังนั้น แทนค่าจะได้

$$\frac{\pi^2}{12} - \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right) - \frac{1}{4} \left( \frac{\pi^2}{6} \right)$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{12} + \frac{\pi^2}{24}$$

$$= \frac{\pi^2}{24} \left( \frac{2 + 1}{24} \right)$$

$$= \frac{\pi^2}{8}$$

2. จากแบบฝึกหัด 2.4 ข้อ 11 จงแสดงว่า

(ก) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(ข) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

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(ก) อนุกรมสูตรเรียลโคลี่ซ์จากแบบมีกัด 2.4 ข้อ 11 (ก) คือ

\( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n - 1)^2} = \frac{\pi^3}{32} \)

(ข) \( \frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} \ldots = \frac{3\pi^2 \sqrt{2}}{108} \)

วิธีทำ อนุกรมสูตรเรียลโคลี่ซ์จากแบบมีกัด 2.4 ข้อ 11 (ก) คือ

\( f(x) = \frac{x^2}{6} - \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} \)

(ก) ถ้าแทนค่า \( x = 0 \) จะเป็นจุดที่ต่อเนื่องของพื้นฐาน \( f(x) \) ดังนั้น

\( f(0) = x(\pi - x) \bigg|_{x=0} = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{(1)}{n^2} \)

\( 0 = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2} \)

หรือ \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \)

(ข) เลือกแทนค่า \( x = \frac{\pi}{2} \) จะเป็นจุดที่ต่อเนื่องของพื้นฐาน \( f(x) \) ดังนั้น

\( f \left( \frac{\pi}{2} \right) = x(\pi - x) \bigg|_{x=\frac{\pi}{2}} = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} \)

\( = \frac{\pi}{2} \left( \pi - \frac{\pi}{2} \right) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \)

\( \frac{\pi^2}{4} - \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \)

\( \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \)

แต่ \( (-1)^{n+1} = (-1)^n \cdot (-1) = (-1)^n \cdot (-1) = (-1)^n \)

และ \( (-1)^{n-1} = (-1)^n \cdot (-1)^{-1} = \frac{(-1)^n}{(-1)} = (-1)^n \)

นั่นคือ \( (-1)^{n+1} = (-1)^{n-1} \) ดังนั้น

\( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \)
(3) ใช้ผลจากแบบฝึกหัด 2.4 ข้อ 11 (ข)

\[ f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{(2n-1)^2} \]

เลือกแทนค่า \( x = \frac{\pi}{2} \) ซึ่งเป็นจุดตัดเย็นของ \( f(x) \) ดังนี้

\[ f\left(\frac{\pi}{2}\right) = x(\pi - x) \mid_{x=\frac{\pi}{2}} = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi/2}{(2n-1)^2} \]

\[ \frac{\pi^2}{4} = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left(n\pi \frac{\pi}{2}\right)}{(2n-1)^2} \]

เพราะว่า

\[ \sin\left(n\pi \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - n\pi\right) = -\cos n\pi = (-1)^n \]

เพราะฉะนั้น

\[ \frac{\pi^2}{4} = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \]

หรือ

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = \frac{\pi^2}{32} \]

เพราะว่า \((-1)^{n-1} = (-1)^n\)

(4) เลือกแทนค่า \( x = \frac{\pi}{4} \) ซึ่งเป็นจุดตัดเย็น ดังนี้

\[ f\left(\frac{\pi}{4}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi/4)}{(2n-1)^2} \]

\[ \frac{\pi}{4} \left(\pi - \frac{\pi}{4}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi/4)}{(2n-1)^2} \]

\[ \frac{3\pi^2}{16} \left(\frac{\pi}{8}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi)}{(2n-1)^2} \]
\[ \frac{3\pi^3}{128} = \sin \frac{\pi}{4} + \sin \frac{3\pi}{4} + \sin \frac{5\pi}{4} + \ldots \]

\[ = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{9\sqrt{2}} - \frac{1}{11\sqrt{2}} + \ldots \]

หรือ \[ \frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{9^3} - \frac{1}{11^3} - \ldots = \frac{3\pi^3\sqrt{2}}{128} \]

3. ใช้ผลจากแบบฝึกหัด 2.4 ข้อ 11 และเอกลักษณ์ของปรารถนาค Fauchet จงแสดงว่า

\[
(n) \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \\
(ข) \quad \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{945}
\]

วิธีทำ  (γ) ใช้ผลจากแบบฝึกหัด 2.4 ข้อ 11 (γ)

\[ x(x - \pi) = \frac{\pi^3}{6} - \frac{\pi}{2} \sum_{n=1}^{\infty} \cos 2nx \]

\[ a_0 = \frac{\pi^3}{3}, \quad a_n = \frac{-2n + (-1)^n}{n^2} \]

จากสูตรเอกลักษณ์ของปรารถนาค Fauchet การนี้ \( f(x) \) นิยามเพียงครึ่งช่วง \((0, \pi)\) คือ

\[ \frac{2}{\pi} \int_{0}^{\pi} |f(x)|^2 \, dx = \frac{1}{2} m^2 + \sum_{n=1}^{\infty} \pi^2 \]

\[ = \frac{2}{\pi} \int_{0}^{\pi} |x(x - \pi)|^2 \, dx = \frac{1}{2} \left( \frac{\pi^3}{3} \right)^2 + \sum_{n=1}^{\infty} \left[ \frac{-2n + (-1)^n}{n^2} \right]^2 \]

\[ = \frac{2}{\pi} \int_{0}^{\pi} (x^2 \pi^2 + 2\pi x^3 + x^4) \, dx = \frac{1}{2} \left( \frac{\pi^3}{3} \right)^2 + \sum_{n=1}^{\infty} \frac{4n + (-1)^n}{n^2} \]

\[ = \left[ \frac{2}{\pi} \left( \frac{x^3}{3} - 2\pi \left( \frac{x^4}{4} + \frac{x^5}{5} \right) \right) \right]_{0}^{\pi} = \pi^4 + \frac{16}{2^4} + 0 + \frac{16}{4^4} + 0 + \frac{16}{6^4} + \ldots \]

\[ = \frac{2}{\pi} \left[ \frac{\pi^3}{3} - \frac{\pi^5}{2} + \frac{\pi^7}{5} \right] = \frac{\pi^4}{18} + \left( \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots \right) \]
\[ \frac{2}{\pi} \left( \frac{10 - 15 + 6}{30} \right) \pi^4 = \frac{\pi^4}{18} + \left( \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots \right) \]

\[ \frac{\pi^4}{15} - \frac{\pi^4}{18} = \left( \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots \right) \]

\[ \left( \frac{65 - 5}{90} \right) \pi^4 = \left( \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots \right) \]

หรือ \[ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90} \]

(ข) ผลจากแบบฝึกหัด 2.4 ข้อ 11 (ข)

\[ x(\pi - x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \sin \left( \frac{2n}{\pi} \right) \frac{x}{n - 1} \]

\[ b_n = \frac{8}{\pi(2n - 1)} \]

ใช้สูตรเอกลักษณ์ของปรารถนาวิถี กรณี \( f(x) \) นิยามเพียงครั้งเดียว \((0, \pi)\) ในเมื่อ \( a_0 = 0 \) และ \( a_n = 0 \) นั่นคือ

\[ \frac{2}{\pi} \int_0^\pi |f(x)|^2 \, dx = \sum_{n=1}^{\infty} b_n^2 \]

แทนค่า \( f(x) = x(\pi - x) \) และ \( b_n = \frac{8}{\pi(2n - 1)} \) จะได้

\[ \frac{2}{\pi} \int_0^\pi |x(\pi - x)|^2 \, dx = \sum_{n=1}^{\infty} \left( \frac{8}{\pi(2n - 1)} \right)^2 \]

ใช้ผลจากข้อ (η) อันถูกวิสัยแล้วขั้นนำจะมีค่าเท่ากับ \( \frac{\pi^4}{15} \)

ดังนั้น

\[ \frac{\pi^4}{15} = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^6} \]

หรือ \[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{15 \times 64} = \frac{\pi^6}{960} \]

กำหนดให้

\[ S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \ldots \]

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\[
= \left( \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \ldots \right) + \left( \frac{1}{1^6} + \frac{1}{4^6} + \frac{1}{6^6} + \ldots \right)
\]
\[
= \frac{\pi^6}{960} + \frac{1}{2^6} \left( \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots \right)
\]
\[
= \frac{\pi^6}{960} + \frac{1}{64} S
\]
\[
\left( 1 - \frac{1}{64} \right) S = \frac{\pi^6}{960}
\]

\[
S = \frac{\pi^6}{960} \left( \frac{64}{63} \right) = \frac{\pi^6}{945}
\]

\[
\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots = \frac{\pi^6}{945}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}
\]

4. จงแสดงว่า

\[
\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots = \frac{\pi^2 - 8}{16}
\]

แนะนำ : ใช้ผลรวมพีซีรี่ยร์โคชีของฟังก์ชัน

\[
sin x = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(1 + \cos nx)}{n^2 - 1} \cos nx ; \quad 0 < x < \pi
\]

วิธีทำ อนุกรมมี

\[
a_0 = \frac{4}{\pi} \text{ และ } a_n = -\frac{2(1 + \cos nx)}{\pi(n^2 - 1)}
\]

สูตรเอกลักษณ์ของปริมาตรมวล ในกรณีที่ \( f(x) \) นิยามเพียงครั้งช่วง \((0, \pi)\) คือ

\[
\frac{2}{\pi} \int_0^\pi \left| f(x) \right|^2 \, dx = \frac{1}{2} \, a_0^2 + \sum_{n=1}^{\infty} a_n^2
\]

\[
\frac{2}{\pi} \int_0^\pi \left( \sin x \right)^2 \, dx = \frac{1}{2} \left( \frac{4}{\pi} \right)^2 + \sum_{n=1}^{\infty} \left| \frac{-2(1 + (-1)^n)}{\pi(n^2 - 1)} \right|^2
\]
\[ \frac{2}{\pi} \int_{0}^{\pi} \left( \frac{1 - \cos 2x}{2} \right) \, dx = \frac{16}{2\pi^2} + \frac{1}{\pi^2} \left[ 0 + \frac{16}{3^2} + 0 + \frac{16}{15^2} + 0 + \frac{16}{35^2} + \ldots \right] \]
\[ \frac{1}{\pi} \left[ x - \frac{\sin 2x}{2} \right]_{0}^{\pi} = \frac{16}{2\pi^2} + \frac{16}{\pi^2} \left[ \frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots \right] \]
\[ \frac{1}{\pi} \left[ \pi - 0 \right] = \frac{16}{2\pi^2} + \frac{16}{\pi^2} \left[ \frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots \right] \]
\[ \frac{16}{2\pi^2} = \frac{16}{\pi^2} \left[ \frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots \right] \]

\[ \frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots = \frac{\pi^2 - 8}{16} \]

5. จงแสดงว่า

\[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{96} \]

\[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{96} \]

วิธีทำ (n) ใช้ทฤษฎีบทเรียกว่า จากตัวอย่างที่ 2.9 ของพหุพันธ์

\[ f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases} \]

ให้ เมื่อ \( a_0 = \frac{-\pi}{2}, \ a_n = \frac{1}{\pi} \left\{ \frac{(-1)^n}{n^2} - 1 \right\} \) และ

\[ b_n = \frac{1 - 2(-1)^n}{n} \]

สูตรผลลัพธ์ของปริมาตรของวัสดุ คือ

\[ \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \]

\[ \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} (-\pi)^2 \, dx + \int_{0}^{\pi} (x)^2 \, dx \right] = \frac{1}{2} \left( -\frac{\pi}{2} \right)^2 + \sum_{n=1}^{\infty} \left\{ \frac{1}{\pi} \left( \frac{(-1)^n}{n^2} - 1 \right) \right\}^2 \]

\[ + \left\{ \frac{1 - 2(-1)^n}{n} \right\}^2 \]

\[ \frac{1}{\pi} \left[ \pi^2 x \int_{-\pi}^{0} \frac{x^2}{3} \, dx \right] = \frac{\pi^2}{8} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{n} - 1 \right)^2 + \sum_{n=1}^{\infty} \left( \frac{1 - 2(-1)^n}{n} \right)^2 \]

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\[
\frac{1}{\pi} \left[ \pi^3 + \frac{\pi^1}{3} \right] = \frac{\pi^2}{8} + \frac{1}{\pi^2} \left[ \frac{4}{1^4} + 0 + \frac{4}{3^4} + 0 + \frac{4}{5^4} + \ldots \right] \\
\quad + \left[ \frac{9}{1^2} + \frac{1}{2^2} + \frac{9}{3^2} + \frac{1}{4^2} + \ldots \right]
\]

\[
\frac{4\pi^2}{3} - \frac{\pi^2}{8} = \frac{4}{\pi^2} \left[ \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots \right] + 9 \left[ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right] \\
\quad + \left[ \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots \right]
\]

\[
\left( \frac{32 - 3}{24} \right) \pi^2 = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} + 9 \left( \frac{\pi^2}{8} \right) \\
\quad + \frac{1}{2^2} \left[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \right]
\]

\[
\frac{29\pi^2}{24} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} + \frac{9\pi^2}{8} + \frac{1}{4} \left( \frac{\pi^2}{6} \right)
\]

\[
\frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{29\pi^2}{24} \cdot \frac{9\pi^2}{8} = \frac{\pi^2}{24}
\]

\[
= \frac{(29 - 27 - 1)\pi^4}{24}
\]

\[
\frac{\pi^4}{24}
\]

หรือ \[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{24} \left( \frac{\pi^2}{4} \right) = \frac{\pi^6}{96}
\]

หมายเหตุ มีการใช้ผลจากแบบฝึกหัด 2.5 ข้อ 1 (n) และ (c)

(ข) ใช้ผลจากแบบฝึกหัด 2.5 ข้อ 3 (ข)

กำหนดให้ \[
S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \ldots
\]

\[
= \left( \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots \right) + \left( \frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \ldots \right)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \frac{1}{2^6} \left( \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots \right)
\]
\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{1}{64} S \\
(1 - \frac{1}{64}) S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} \\
\frac{75}{64} S = \frac{\pi^6}{945} \\
S = \frac{\pi^6}{945} \\
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{63}{64} \left( \frac{\pi^6}{945} \right) = \frac{\pi^6}{960}
\]
บทที่ 2.6

จากข้อ 1 ถึงข้อ 6 จะหาอนุกรมฟูร์รีย์ในรูปเชิงซ้อนของฟังก์ชันกับ ซึ่งนิยามใน
หนึ่งคือ

1. \[ f(x) = \begin{cases} 1 & ; 0 < x < 1 \\ 0 & ; 1 < x < 2 \end{cases} \]

วิธีทำ

สูตรอนุกรมฟูร์รีย์ในรูปเชิงซ้อน คือ

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \]

ในนี้ \( c_n = \frac{1}{2\ell} \int_{c}^{c+2\ell} f(x) e^{-inx} \, dx \)

เฉพาะค่า \( \ell = 2\pi \) หรือ \( \ell = \frac{2\pi}{n \pi} = \frac{2\pi}{n \pi} = 0 \)

ดังนั้น

\[
c_n = \frac{1}{2\ell} \int_{0}^{2\ell} f(x) e^{-inx} \, dx
\]

\[
= \frac{1}{2} \left[ \int_{0}^{1} (1) e^{-inx} \, dx + \int_{1}^{2} (0) e^{-inx} \, dx \right]
\]

\[
= \frac{1}{2} \left[ \frac{-e^{-inx}}{in\pi} \right]_{0}^{1} + 0
\]

\[
= \frac{1}{2} \left[ - \frac{e^{-inx}}{in\pi} - 1 \right]
\]

\[
= \frac{1}{2in\pi} (1 - e^{-inx})
\]

แทนค่าในสูตรอนุกรมฟูร์รีย์ในรูปเชิงซ้อน

\[
f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2in\pi} (1 - e^{inx}) e^{inx}
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{i}{2\pi} (e^{-inx} - 1) e^{inx}
\]

เฉพาะค่า \( e^{-inx} = \cos nx - i \sin nx = \cos nx = (-1)^n \) เพราะฉะนั้น

\[
f(x) = \sum_{n=-\infty}^{\infty} \frac{i}{2\pi} \left[ (-1)^n - 1 \right] e^{inx}
\]
2. \( f(x) = \begin{cases} 1 & ; \quad 0 < x < 1 \\ -1 & ; \quad 1 < x < 2 \end{cases} \)

วิธีทํา สูตรอนุกรมฟูร์เรีย ในการปรับเชนคือ

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \]

ในมือ \( c_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) e^{-inx} \, dx \)

เพราะว่า 1 เท่านี้ = 2\( \pi \) = 2 เพื่อแสดงค่า \( r = 1 \) เลือก \( c = 0 \)

\[
c_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) e^{-inx} \, dx \]

\[
= \frac{1}{2} \left[ \int_{0}^{1} e^{inx} \, dx + \int_{1}^{2} (e^{inx} - e^{-inx}) \, dx \right] \]

\[
= \frac{1}{2} \left[ \frac{1}{in\pi} \left[ e^{inx} \right]_{0}^{1} + \frac{1}{in\pi} \left[ e^{inx} - e^{-inx} \right]_{1}^{0} \right] \]

เพราะว่า \( e^{inx} = \cos{n\pi} - i\sin{n\pi} = \cos{n\pi} = (-1)^n \)

และ \( e^{inx} = \cos{2n\pi} - i\sin{2n\pi} = 1 \)

เพราะฉะนั้น แทนค่าจะได้

\[
c_n = \frac{1}{2in\pi} \left[ - \left\{ (-1)^n - 1 \right\} + \left\{ 1 - (-1)^n \right\} \right] \]

\[
= \frac{1}{2in\pi} \left[ \left\{ (-1)^n - 1 \right\} - \left\{ (-1)^n - 1 \right\} \right] \]

\[
= \frac{1}{2n\pi} \left\{ (-1)^n - 1 \right\} \]

แทนค่า \( c_n \) ในสูตรอนุกรมฟูร์เรีย ของฟังก์ชัน

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \]

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3. \( f(x) = x ; \ 0 < x < 1 \)

วิธีทำ  สูตรอนุกรมฟูเรียร์ ในรูปเรียงคือ

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}/l
\]

จากสูตร \( c_n = \frac{1}{2l} \int_{c}^{c+2l} f(x) e^{-inx}/l \ dx \)

เพราะว่า \( \delta = 2l = 1 \) เพราะฉนั้น \( \ell = \frac{1}{2} \) เลือกแทนค่า \( c = 0 \)
ดังนั้น

\[
c'' = \frac{1}{2} \left( \frac{1}{2} \right) \left[ \int_{0}^{1} e^{-2inx} dx \right]
\]

\[
= \int_{0}^{1} x e^{-2inx} dx
\]

อินทิเกเตตที่ยื่นส่วน ให้ \( u = x \) และ \( dv = e^{-2inx} \) จะได้

\[
\int_{0}^{1} x e^{-2inx} dx = x \left( \frac{e^{-2inx}}{2inx} \right) \bigg|_{0}^{1} - \int_{0}^{1} \left( \frac{-e^{-2inx}}{2inx} \right) dx
\]

\[
= \frac{-1}{2inx} \left( e^{-2inx} \right)_{0}^{1} + \frac{1}{4n^{2} \pi^{2}} e^{-2inx} \bigg|_{0}^{1}
\]

\[
= -\frac{e^{-2inx}}{2inx} + \frac{1}{4n^{2} \pi^{2}} (e^{-2inx} - 1)
\]

ดังนั้น

\[
c_n = \frac{1}{4n^{2} \pi^{2}} (e^{-2inx} - 1) = \frac{e^{-2inx}}{2inx}
\]

เพราะว่า \( e^{-2inx} = \cos 2nn - i \sin 2nn = 1 \)
เพราะฉนั้น

\[
c_n = \frac{1}{4n^{2} \pi^{2}} \left[ 1 - 1 \right] - \frac{1}{2inx} \]

\[
= 0 + \frac{i}{2nn} \]

\( i \)

2nn

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แทนค่า $c_n$ ในสูตรอนุกรมฟูเรียร์ในรูปเชิงซ้อน

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{i}{2\pi n} e^{inx}$$

หรือ $x = \frac{i}{2\pi} \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n}$

4. $f(x) = x$ ; $-1 < x < 1$

วิธีทำ สูตรอนุกรมฟูเรียร์ในรูปเชิงซ้อน คือ

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

จากสูตร $c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-inx} dx$

เพราะว่า $\ell = 2$ เพราะฉะนั้น $\ell = 1$ แทนค่าใน $c_n$ จะได้

$$c_n = \frac{1}{2\ell} \left[ \frac{1}{i} \int_{-\ell}^{\ell} (x) e^{-inx} dx \right]$$

$$= \frac{1}{2} \left[ \frac{-1}{in\pi} \right] \left[ \left( \frac{e^{inx}}{in\pi} \right) \right]_{-1}^{1} - \frac{1}{2} \left( \frac{-1}{in\pi} \right)$$

$$= \frac{1}{2} \left[ \frac{-1}{in\pi} \right] \left( \frac{e^{in}-e^{-in}}{in\pi} \right)$$

$$= \frac{1}{2} \left[ \left( -1 \right)^n + \left( -1 \right)^n \right] + \frac{1}{n^2\pi^2} \left( e^{in} - e^{-in} \right)$$

$$= \frac{-(-1)^n}{in\pi} + \frac{i(-1)^n}{n\pi}$$

แทนค่า $c_n$ ในสูตรอนุกรมฟูเรียร์รูปเชิงซ้อน

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{i(-1)^n}{n\pi} e^{inx}$$


\[ f(x) = \cos x \quad \frac{-\pi}{2} < x < \frac{\pi}{2} \]

วิธีทำ สูตรอนุกรมฟูเรียร์ในรูปเชิงซ้อนคือ

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}\]

จากสูตร \( c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-inx} \, dx \)

เพราะว่า \( \ell = 2\ell = \pi \) เพราะฉะนั้น \( \ell = \frac{\pi}{2} \) แทนค่าใน \( c_n \) จะได้

\[
c_n = \frac{1}{2} \left( \frac{\pi}{2} \right) \int_{-\pi/2}^{\pi/2} \cos x e^{-2inx} \, dx
\]

เพราะว่า \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \) เพราะฉะนั้น

\[
c_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left( \frac{e^{ix} + e^{-ix}}{2} \right) e^{-2inx} \, dx
\]

\[
= \frac{1}{2\pi} \left[ \int_{-\pi/2}^{\pi/2} e^{ix} e^{-2inx} \, dx + \int_{-\pi/2}^{\pi/2} e^{-ix} e^{-2inx} \, dx \right]
\]

\[
= \frac{1}{2\pi} \left[ \int_{-\pi/2}^{\pi/2} e^{i(1-2n)x} \, dx + \int_{-\pi/2}^{\pi/2} e^{i(1+2n)x} \, dx \right]
\]

\[
= \frac{1}{2\pi} \left[ \frac{e^{i(1-2n)x} - e^{i\pi/2}}{i(1 - 2n)} - \frac{e^{i(1+2n)x} - e^{-i\pi/2}}{i(1 + 2n)} \right]
\]

\[
= \frac{1}{2\pi} \left[ \left\{ \frac{e^{i(1-2n)x/2} - e^{i(2n-1)x/2}}{i(1 - 2n)} \right\} - \left\{ \frac{e^{i(1+2n)x/2} - e^{-i(1+2n)x/2}}{i(1 + 2n)} \right\} \right]
\]

เพราะว่า

\[
e^{i(1-2n)x/2} = \cos (1 - 2n) \frac{\pi}{2} + i \sin (1 - 2n) \frac{\pi}{2}
\]
และ \( \cos \left( l \ \frac{\pi}{2} \right) = \cos \left( \frac{\pi}{2} - n\pi \right) = \sin n\pi = 0 \)

\( \sin \left( l - 2n \right) \frac{\pi}{2} = \sin \left( \frac{\pi}{2} - n\pi \right) = \cos n\pi = ( -1 )^n \)

เพราะฉนั้น

\[ e^{i(l - 2n \pi/2)} = 0 + i(-1)^n = i(-1)^n \]

ในทั่วทั้งสิ้นเพื่อความเข้าใจ

\[ e^{i(l - 2n \pi/2)} = e^{-i(l - 2n \pi/2)} \]

\[ = \cos \left( l \ 2n \right) \frac{\pi}{2} + i \sin \left( l \ 2n \right) \frac{\pi}{2} \]

\[ = 0 - i(-1)^n \]

\[ = -i(-1)^n \]

\[ \text{และ} \quad e^{-i(l + 2n \pi/2)} = e^{i(l + 2n \pi/2)} \]

\[ = \cos \left( l + 2n \right) \frac{\pi}{2} + i \sin \left( l + 2n \right) \frac{\pi}{2} \]

\[ = \cos \left( \frac{\pi}{2} + n\pi \right) + i \sin \left( \frac{\pi}{2} + n\pi \right) \]

\[ = -\sin n\pi + i \cos n\pi \]

\[ = 0 + i(-1)^n \]

\[ = i(-1)^n \]

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แทนค่า (2), (3), (4) และ (5) ใน (1) จะได้

\[ c_n = \frac{1}{2\pi} \left[ \frac{i(-1)^n + i(-1)^n}{1 - 2n} - \frac{-i(-1)^n - i(-1)^n}{i(1 + 2n)} \right] \]

\[ = \frac{1}{2\pi} \left[ \frac{2i(-1)^n + 2i(-1)^n}{i(1 - 2n)} \right] \]

\[ = \frac{(-1)^n}{\pi} \left[ \frac{1}{1 - 2n} + \frac{1}{1 + 2n} \right] \]

\[ = \frac{\pi}{\pi} \left[ \frac{2}{4n^2} \right] \]

แทนค่า \( c_n \) ในสูตรอนุกรมฟูเรียร์ในรูปซีซั่น

\[ f(x) = \sum_{n=-\infty}^{\infty} \frac{2(-1)^n}{\pi(1 - 4n^2)} e^{2inx} \]

หรือ \( \cos x = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1 - 4n^2} e^{2inx} \)

6. \( f(x) = \sin x ; \quad 0 < x < \pi \)

วิธีทำ สูตรอนุกรมฟูเรียร์แบบซีซั่น คือ

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \]

จากสูตร \( c_n = \frac{1}{2\ell} \left[ \int_{c}^{c+2\ell} f(x) e^{-inx} dx \right] \)

เพราะว่า 1 คาบ = 2\ell = \pi เพราะฉะนั้น \( \ell = \frac{\pi}{2} \) เลือก \( c = 0 \) แทนค่าใน \( c_n \) จะได้

\[ c_n = \frac{1}{2\ell} \left( \frac{\pi}{2} \right) \int_{0}^{\pi} \left( \frac{e^{i\pi} - e^{-i\pi}}{2i} \right) e^{-inx} dx \]

\[ = \frac{1}{\pi} \int_{0}^{\pi} \left( \frac{e^{ix} - e^{-ix}}{2i} \right) e^{-inx} dx \]

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\[
\begin{align*}
\int_0^\pi e^{ix} - e^{-2inx} \, dx &= -\frac{1}{2\pi i} \left[ \int_0^\pi e^{(1+2n)x} \, dx - \int_0^\pi e^{(1+2n)x} \, dx \right] \\
&= \frac{1}{2\pi i} \left[ \int_0^\pi \frac{e^{i(1-2nx)}}{i(1-2n)} \, dx + \int_0^\pi \frac{e^{i(1+2nx)}}{i(1+2n)} \, dx \right] \\
&= \frac{-1}{2\pi} \left\{ \left\{ \frac{e^{i-2nx} - 1}{1 - 2n} \right\} + \left\{ \frac{e^{i+2nx} - 1}{1 + 2n} \right\} \right\}
\end{align*}
\]

พจน์ที่ 1
\[
e^{i\pi} 2n\pi = \cos (1 - 2n)\pi + i \sin (1 - 2n)\pi
\]
\[
= \cos (7\pi - 2n\pi) + i \sin (\pi - 2n\pi)
\]
\[
-\cos 2n\pi + i \sin 2n\pi
\]
\[
= -(1) + 0
\]
\[
= -1
\]

และ
\[
e^{i(1+2n)\pi} = \cos (1 + 2n)\pi + i \sin (1 + 2n)\pi
\]
\[
= \cos (\pi + 2n\pi) + i \sin (\pi + 2n\pi)
\]
\[
-\cos 2n\pi + i \sin 2n\pi
\]
\[
= -(1) \quad i(0)
\]
\[
= -1
\]

พจน์ที่ 2
\[
e_R = \frac{-1}{2\pi} \left\{ \left\{ \frac{1 - 1}{1 - 2n} \right\} + \left\{ \frac{-1 - 1}{1 + 2n} \right\} \right\}
\]
\[
= \frac{-1}{2\pi} \left\{ \frac{-2}{1 - 2n} + \frac{-2}{1 + 2n} \right\}
\]
\[
= \frac{1}{\pi} \left\{ \frac{1 + 2n + 1 - 2n}{1 - 4n^2} \right\}
\]
\[
= \frac{2}{\pi(1 - 4n^2)}
\]
แทนค่า $c_n$ ในสูตรอนุกรมฟูเรียร์แบบเชิงซ้อน

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1 - 4n^2)} e^{inx}$$

หรือ $\sin x = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{1 - 4n^2} e^{inx}$